APPROXIMATIONS AND ERRORS

In science and mathematics, exact values are often impossible to obtain, either because of measurement limitations or because some numbers (like π or $\sqrt{2}$) cannot be written exactly as terminating decimals. Therefore, we often work with approximations.
In this chapter, we learn how to round numbers correctly, how to quantify the error introduced by an approximation, how measurement accuracy is linked to the smallest division on a scale, and how to determine bounds (an interval of possible true values).

A ROUNDING

Definition Rounding Rules

We can round numbers in two main ways:

- **Decimal places (dp):** the number of digits kept *after* the decimal point.
- **Significant figures (sf):** the number of digits kept starting from the *first non-zero* digit.

Rounding rule (to the nearest): look at the first digit that will be *removed*. If it is $0, 1, 2, 3, 4$, round down; if it is $5, 6, 7, 8, 9$, round up.

Ex: Consider the number $N = 1204.5678$.

- Rounded to **2 decimal places (2 dp)**: 1204.57 (the 3rd decimal digit is 7, so we round up).
- Rounded to **3 significant figures (3 sf)**: 1.20×10^3 (the next digit is 4, so we round down).
(Writing 1200 is acceptable here, but 1.20×10^3 makes the "3 sf" explicit.)

Definition Estimation

Estimation means finding a value that is close enough to the correct answer using simpler calculations. It is useful for checking whether calculator results are reasonable (order of magnitude, size, and plausibility).

Method Estimation Strategy

A common strategy is:

1. Round each number to **1 significant figure**.
2. Perform the calculation with the rounded numbers.
3. Interpret the result as an approximate check (not an exact answer).

Ex: A restaurant meal costs \$9.90 per person. Estimate the total cost for 5 people.

Answer: To estimate, round to **1 significant figure**:

$$9.90 \approx 10 \quad \text{and} \quad 5 \approx 5.$$

So the estimated total cost is:

$$10 \times 5 = 50.$$

(Exact total: $9.90 \times 5 = 49.50$, so the estimate is close.)

B ERROR FORMULAS

Definition Absolute, Relative and Percentage Error

Let V_{exact} be the true value and V_{approx} be the approximated value.

- The **signed error** is $e = V_{\text{approx}} - V_{\text{exact}}$.
- The **absolute error** is $|e| = |V_{\text{approx}} - V_{\text{exact}}|$.
- The **relative error** (when $V_{\text{exact}} \neq 0$) is

$$r = \left| \frac{V_{\text{approx}} - V_{\text{exact}}}{V_{\text{exact}}} \right|.$$

- The **percentage error** is

$$\epsilon = r \times 100\% = \left| \frac{V_{\text{approx}} - V_{\text{exact}}}{V_{\text{exact}}} \right| \times 100\%.$$

Ex: The estimated length of a table is 150 cm, while the actual length is 145 cm. Calculate the absolute error and the percentage error.

Answer:

- Absolute error: $|150 - 145| = 5$ cm.
- Percentage error:

$$\left| \frac{150 - 145}{145} \right| \times 100\% \approx 3.45\%.$$

C MEASUREMENT ACCURACY

When we take measurements, we usually read a scale (a ruler, a thermometer, a balance, ...). Often, the true value lies between two marks. We record the closest mark, so the recorded value is an approximation.
If the smallest division on the scale is u, then the measurement can be wrong by up to half of this division, i.e. by $\pm\frac{u}{2}$.

Definition Accuracy Rule

If the smallest division on the scale is u, then:

$$\text{accuracy} = \pm\frac{u}{2}.$$

This is sometimes called the **reading error** or the **absolute uncertainty**.

Ex 1: A ruler has markings every 1 cm (so $u = 1$ cm). A pencil is measured to be 14 cm.

1. Determine the accuracy of the measurement.
2. State the interval within which the true length lies.

Answer:

1. The accuracy is $\pm\frac{1}{2} \times 1 \text{ cm} = \pm 0.5$ cm.
2. The true length L is in the interval $[13.5,\ 14.5)$ cm, i.e. $13.5 \leq L < 14.5$ cm.

Note: If the ruler had millimetre marks ($u = 1$ mm $= 0.1$ cm), the accuracy would be ± 0.5 mm $= \pm 0.05$ cm.

D BOUNDS

When a value is rounded to the nearest unit u, the exact value lies in:

$$\text{Rounded Value} - \frac{u}{2} \leq x < \text{Rounded Value} + \frac{u}{2}.$$

(Examples: nearest cm $\Rightarrow u = 1$; nearest 0.1 $\Rightarrow u = 0.1$; rounded to 2 dp $\Rightarrow u = 0.01$.)

Definition Lower and Upper Bounds

For a rounded value:

- The **Lower Bound** (LB) is the smallest possible value.
- The **Upper Bound** (UB) is the greatest possible value (usually not included), so we often write "<" for the upper bound.

Ex 2: The length of a book is given as $L = 12$ cm, rounded to the nearest cm ($u = 1$).

1. Calculate the lower bound and the upper bound of the length.
2. Write the inequality representing the possible values for the true length.

Answer:

1. **Bounds:**
   - Lower bound: $12 - 0.5 = 11.5$ cm.
   - Upper bound: $12 + 0.5 = 12.5$ cm.
2. **Inequality:** Therefore: $11.5 \leq L < 12.5$.

Method Calculating Maximum and Minimum Values

When performing calculations with rounded values (area, perimeter, speed, ...), errors propagate.
Let A and B be two **positive** quantities with bounds $A \in [A_{\text{min}}, A_{\text{max}}]$ and $B \in [B_{\text{min}}, B_{\text{max}}]$ (with $B_{\text{min}} > 0$ if dividing by B).

Operation	Maximum Value	Minimum Value
$A + B$	$A_{\text{max}} + B_{\text{max}}$	$A_{\text{min}} + B_{\text{min}}$
$A - B$	$A_{\text{max}} - B_{\text{min}}$	$A_{\text{min}} - B_{\text{max}}$
$A \times B$	$A_{\text{max}} \times B_{\text{max}}$	$A_{\text{min}} \times B_{\text{min}}$
$A \div B$	$A_{\text{max}} \div B_{\text{min}}$	$A_{\text{min}} \div B_{\text{max}}$

Ex: A rectangle has length $l = 10$ cm and width $w = 5$ cm, both measured to the nearest cm. Calculate the maximum and minimum possible area.

Answer:

- **Step 1: Determine the bounds.**
  Since the measurements are to the nearest 1 cm, the absolute error is ± 0.5 cm.
  - Length l: Lower bound $= 9.5$, Upper bound $= 10.5$.
  - Width w: Lower bound $= 4.5$, Upper bound $= 5.5$.
- **Step 2: Calculate the areas.**
  - **Maximum Area** (using upper bounds):

$$A_{\text{max}} = 10.5 \times 5.5 = 57.75 \text{ cm}^2$$

  - **Minimum Area** (using lower bounds):

$$A_{\text{min}} = 9.5 \times 4.5 = 42.75 \text{ cm}^2$$