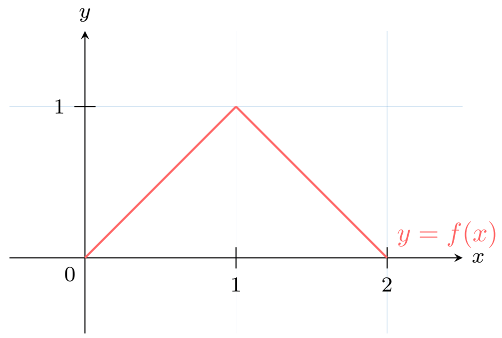
CONTINUOUS RANDOM VARIABLES

A DEFINITIONS

A.1 PROBABILITY DENSITY FUNCTION

A.1.1 CALCULATING PROBABILITIES UNDER THE CURVE

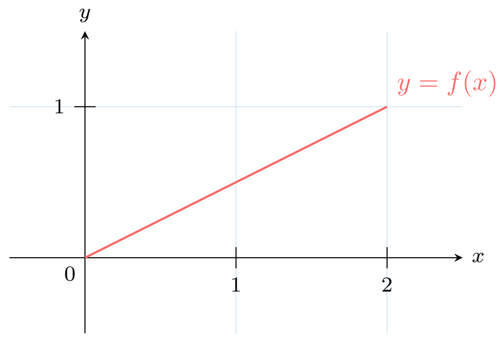
Ex 1: Suppose X represents the time (in hours) a device operates before needing maintenance, with values on $[0, 2]$, and its probability density function $f(x)$ is shown in the graph below.



Using the graph, estimate the probability that the device operates for 1 hour or less.

$$P(0 \leq X \leq 1) = \square$$

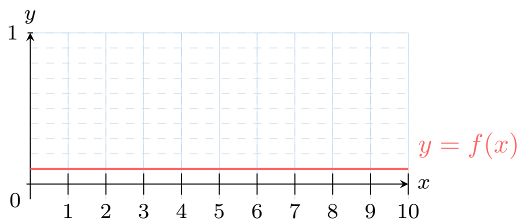
Ex 2: Suppose X represents the waiting time (in minutes) for a bus, with values on $[0, 2]$, and its probability density function is shown in the graph below.



Using the graph, estimate the probability that the waiting time is less than 1 minute.

$$P(X < 1) = \square$$

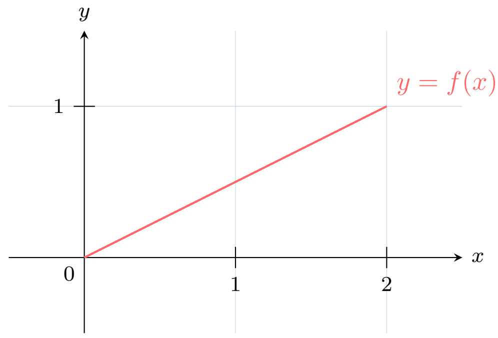
Ex 3: Suppose X represents the waiting time (in minutes) for a bus, which follows a uniform distribution over $[0, 10]$, and its probability density function $f(x)$ is shown in the graph below.



Using the graph, estimate the probability that the waiting time is 4 minutes or less.

$$P(0 \leq X \leq 4) = \square$$

Ex 4: Suppose X represents the waiting time (in minutes) for a bus, with values on $[0, 2]$, and its probability density function is given by $f(x) = \frac{x}{2}$, as shown in the graph below.

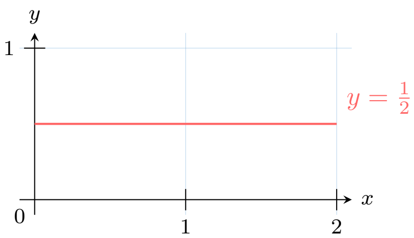


Using the graph, estimate the probability that the waiting time is more than 1 minute.

$$P(X > 1) = \square$$

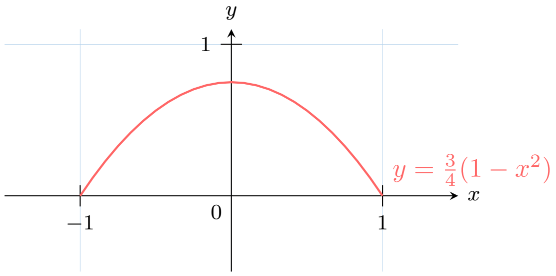
A.1.2 VERIFYING THAT $f(x)$ IS A PROBABILITY DENSITY FUNCTION

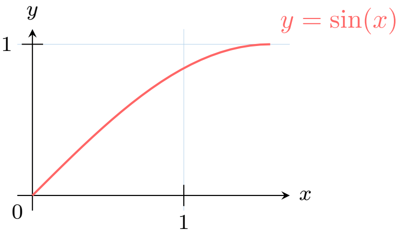
Ex 5: Consider the function $f(x) = \frac{1}{2}$, defined on the interval $[0, 2]$.



Verify that $f(x)$ is a probability density function on the interval $[0, 2]$.

Ex 6: Consider the function $f(x) = \frac{3}{4}(1 - x^2)$, defined on the interval $[-1, 1]$.

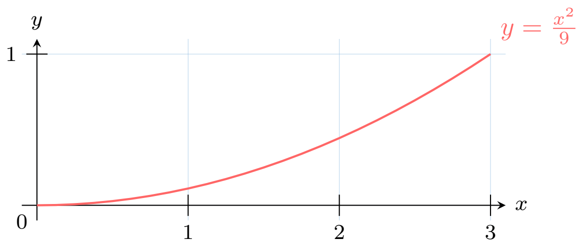




Verify that $f(x)$ is a probability density function on the interval $[-1, 1]$.

Verify that $f(x)$ is a probability density function on the interval $[0, \frac{\pi}{2}]$.

Ex 7: Consider the function $f(x) = \frac{x^2}{9}$, defined on the interval $[0, 3]$.



Verify that $f(x)$ is a probability density function on the interval $[0, 3]$.

Ex 8: Consider the function $f(x) = \sin(x)$, defined on the interval $[0, \frac{\pi}{2}]$.

A.1.3 NORMALIZING A PROBABILITY DENSITY FUNCTION

Ex 9: Consider the function $f(x) = a$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 2]$.

Ex 10: Consider the function $f(x) = ax^3$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 2]$.

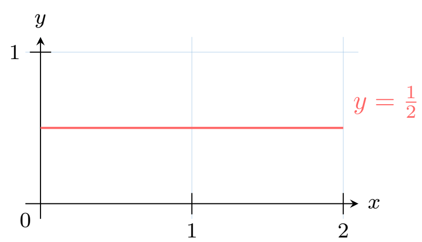
Ex 11: Consider the function $f(x) = a\frac{1}{x}$.

Find the value of a such that $f(x)$ is a probability density function on the interval $[1, 2]$.

Ex 12: Consider the function $f(x) = a\sqrt{x}$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 4]$.

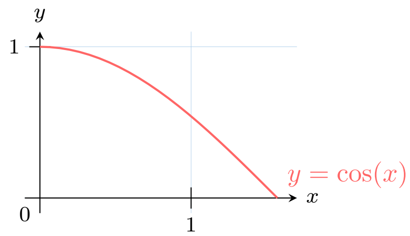
A.1.4 FINDING A PROBABILITY

Ex 13: The random variable X has the density $f(x) = \frac{1}{2}$, on the interval $[0, 2]$.



Find $P(\frac{1}{2} \leq X \leq \frac{3}{4})$.

Ex 14: The random variable X has the density $f(x) = \cos(x)$, on the interval $[0, \frac{\pi}{2}]$.



Find $P(0 \leq X \leq \frac{\pi}{4})$.

Ex 15: The random variable X has the density $f(x) = \frac{1}{\ln(2)x}$, on the interval $[1, 2]$.
Find $P\left(1 \leq X \leq \frac{3}{2}\right)$.

Ex 16: The random variable X has the density $f(x) = \frac{x}{2}$, on the interval $[0, 2]$.

Find $P\left(1 \leq X \leq 2\right)$.

A.2 EXPECTATION

A.2.1 CALCULATING AN EXPECTATION

Ex 17: The random variable X has the density $f(x) = \frac{1}{2}$, on the interval $[0, 2]$. Calculate $E(X)$.

Ex 18: The random variable X has the density $f(x) = \frac{1}{\ln(2)x}$, on the interval $[1, 2]$. Calculate $E(X)$.

Ex 19: The random variable X has the density $f(x) = \frac{x}{2}$, on the interval $[0, 2]$. Calculate $E(X)$.

Ex 20: The random variable X has the density $f(x) = \frac{2}{x^2}$, on the interval $[1, 2]$. Calculate $E(X)$.

A.3 VARIANCE

A.3.1 CALCULATING A VARIANCE

Ex 21: The random variable X with values on $[-1, 1]$ has density $f(x) = \frac{1}{2}$.
Find $V(X)$.

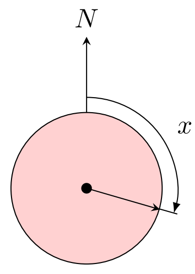
Ex 22: The random variable X with values on $[0,2]$ has density $f(x)=\frac{x}{2}$.
Find $V(X)$.

Ex 23: The random variable X with values on $[1,2]$ has density $f(x)=\frac{2}{x^2}$.
Find $V(X)$.

A.4 CONTINUOUS UNIFORM DISTRIBUTION

A.4.1 EXPLORING THE CONTINUOUS UNIFORM DISTRIBUTION

Ex 24: Consider a random experiment where a spinner is rotated, and the continuous random variable X represents the angle spun, measured in degrees, over the interval $[0,360]$.



1. Determine the probability density function of X.
2. Calculate $P(90 \leq X \leq 180)$.
3. Calculate $P(X \geq 60)$.
4. Calculate the expected value $E(X)$.

Ex 25: Consider a scenario where the continuous random variable X represents the waiting time at a bus stop, uniformly distributed over the interval $[0, 10]$ minutes.

1. Determine the probability density function of X.
2. Calculate $P(X \leq 8)$.
3. Calculate the expected value $E(X)$.

Ex 26: Let X be a continuous random variable following a continuous uniform distribution on $[a, b]$.
Prove that for all $c, d \in [a, b]$,

$$P(c \leq X \leq d) = \frac{d-c}{b-a}.$$

Ex 27: Let X be a continuous random variable following a continuous uniform distribution on $[a, b]$.
Prove that the expected value of X is:

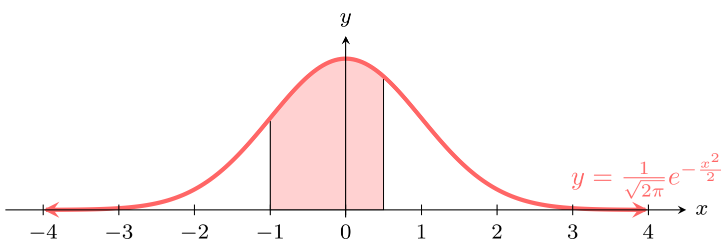
$$E(X) = \frac{a+b}{2}.$$

B NORMAL DISTRIBUTION

B.1 STANDARD NORMAL DISTRIBUTION

B.1.1 FINDING A PROBABILITY FROM AN AREA

MCQ 28: The random variable X follows a standard normal distribution.

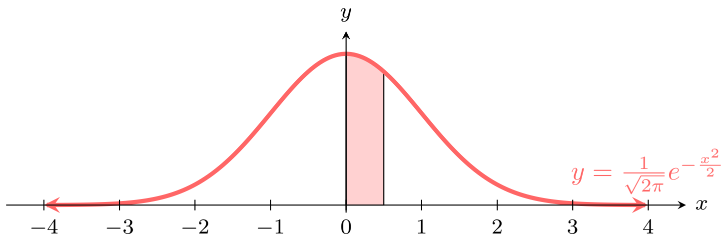


Find the probability corresponding to the red area.
Choose the one correct answer:

- ☐ $P(0 \leq X \leq 0.5)$
- ☐ $P(-1 \leq X \leq 0.5)$
- ☐ $P(X \leq 0.5)$
- ☐ $P(X \geq 1)$

☐ $P(X > -0.5)$

MCQ 29: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.

Choose the one correct answer:

☐ $P(0 \leq X \leq 0.5)$

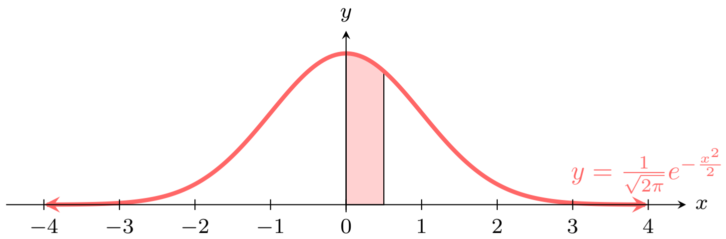
☐ $P(-1 \leq X \leq 0.5)$

☐ $P(X \leq 0.5)$

☐ $P(X \geq 1)$

☐ $P(X > -0.5)$

MCQ 30: The random variable X follow a standard normal distribution.



Find the probability corresponding to the red area.

Choose the one correct answer:

☐ $P(0 \leqslant X \leqslant 0.5)$

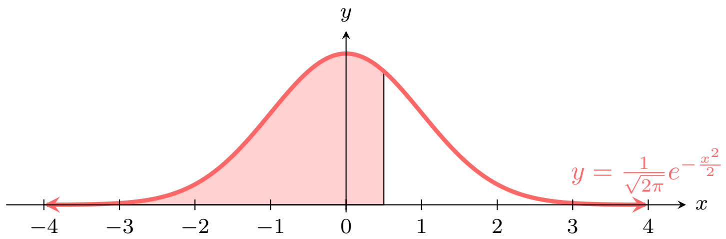
☐ $P(-1 \leqslant X \leqslant 0.5)$

☐ $P(X \leqslant 0.5)$

☐ $P(X \geqslant 1)$

☐ $P(X > -0.5)$

MCQ 31: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.

Choose the one correct answer:

☐ $P(0 \leq X \leq 0.5)$

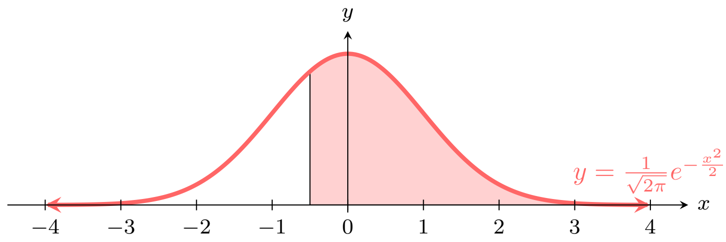
☐ $P(-1 \leq X \leq 0.5)$

☐ $P(X \leq 0.5)$

☐ $P(X \geq 1)$

☐ $P(X > -0.5)$

MCQ 32: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.

Choose the one correct answer:

☐ $P(0 \leq X \leq 0.5)$

☐ $P(-1 \leq X \leq 0.5)$

☐ $P(X \leq 0.5)$

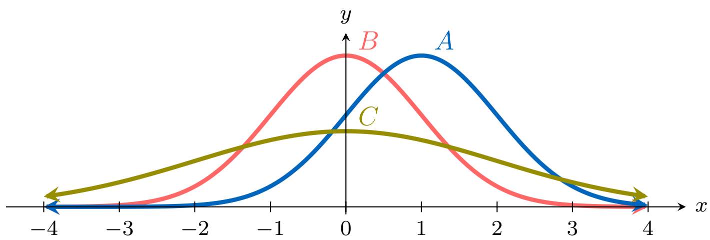
☐ $P(X \geq 1)$

☐ $P(X > -0.5)$

B.2 NORMAL DISTRIBUTION

B.2.1 FINDING THE NORMAL DISTRIBUTION

MCQ 33: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.



Identify which normal distribution has a mean of 1 and a standard deviation of 1.

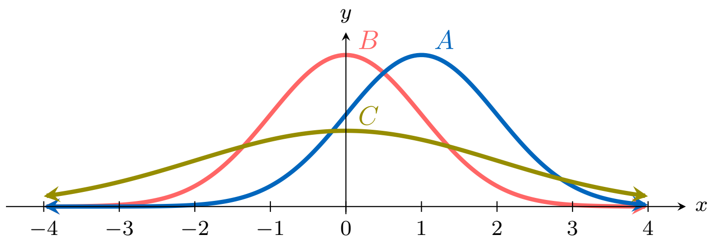
Choose the one correct answer:

☐ Distribution A

☐ Distribution B

☐ Distribution C

MCQ 34: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.

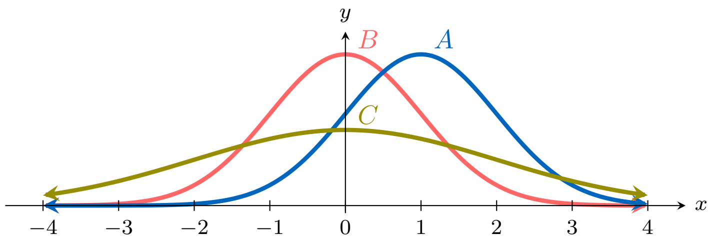


Identify which normal distribution has a mean of 0 and a standard deviation of 1.

Choose the one correct answer:

- ☐ Distribution A
- ☐ Distribution B
- ☐ Distribution C

MCQ 35: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.



Identify which normal distribution has a mean of 0 and a standard deviation of 2.

Choose the one correct answer:

- ☐ Distribution A
- ☐ Distribution B
- ☐ Distribution C

B.2.2 FINDING VALUES USING THE MEAN AND STANDARD DEVIATION

Ex 36: The height of one-year-old babies is normally distributed with a mean of 75 cm and a standard deviation of 3 cm. For medical purposes, a doctor needs to determine the height that corresponds to one standard deviation above the mean.

☐ cm

Ex 37: In a gas at thermal equilibrium, the velocities of particles follow a normal distribution with a mean velocity of 500 m/s and a standard deviation of 100 m/s. A physicist wants to calculate the velocity that corresponds to one standard deviation below the mean.

☐ m/s

Ex 38: The weight of adult women is normally distributed with a mean of 65 kg and a standard deviation of 5 kg. For a health study, a researcher needs to determine the weight that corresponds to two standard deviations above the mean.

☐ kg

Ex 39: The final exam scores in a math course are normally distributed with a mean of 70 points and a standard deviation of 8 points. A teacher wants to identify students who scored one standard deviation below the mean.

☐ points

B.2.3 EXPLORING EVERYDAY STATISTICS

Ex 40: Height and weight are key measurements for tracking a child's development. The World Health Organization assesses child development by comparing the weights of children of the same height and gender. In 2009, the weights of all 80 cm girls in a reference population were normally distributed with a mean of 10.2 kg and a standard deviation of 0.8 kg.
Using this information, calculate the following probabilities or values for the weights of 80 cm girls:

1. The percentage of girls with weights between 10.2 kg and 11 kg.

   ☐ %

2. The percentage of girls with weights between 10.2 kg and 11.8 kg.

   ☐ %

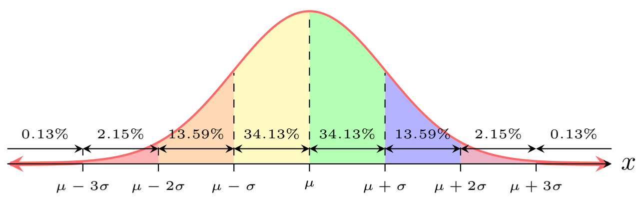
3. The percentage of girls with weights greater than 9.4 kg.

   ☐ %

4. In 2010, if there were 545 girls who were 80 cm tall, estimate the number of girls with weights between 9.4 kg and 11 kg (round to the nearest integer).

   ☐ girls

For a normal distribution, the coverage probabilities are illustrated below:



Ex 41: Exam scores are a key measure for evaluating student performance. A national education board assesses student achievement by analyzing scores from a standardized test. In 2023, the scores of all students in a particular grade were normally distributed with a mean of 75 points and a standard deviation of 5 points.
Using this information, calculate the following probabilities or values for the students' scores:

1. The percentage of students with scores between 70 and 75 points.

   ☐ %

2. The percentage of students with scores between 65 and 75 points.

   ☐ %

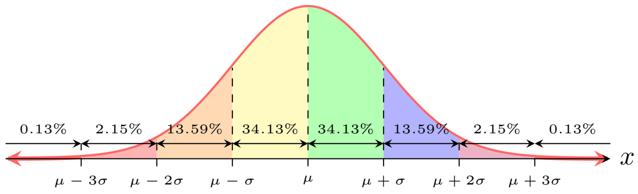
3. The percentage of students with scores less than 80 points.

   ☐ %

4. In 2024, if there were 600 students in this grade, estimate the number of students with scores between 70 and 85 points (round to the nearest integer).

☐ students

For a normal distribution, the coverage probabilities are illustrated below:



Ex 42: Intelligence Quotient (IQ) scores are widely used to measure cognitive ability. A psychological research institute analyzes IQ scores to understand population intelligence distributions. In 2023, the IQ scores of a large adult population were normally distributed with a mean of 100 and a standard deviation of 15.
Using this information, calculate the following probabilities or values for the IQ scores:

1. The percentage of adults with IQ scores between 85 and 100.

☐ %

2. The percentage of adults with IQ scores between 70 and 100.

☐ %

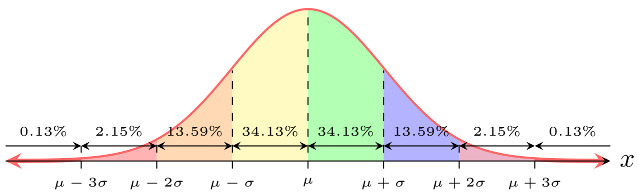
3. The percentage of adults with IQ scores less than 115.

☐ %

4. In 2024, if there were 800 adults in this population, estimate the number of adults with IQ scores greater than 130 (round to the nearest integer).

☐ adults

For a normal distribution, the coverage probabilities are illustrated below:



Ex 43: Daily screen time is a critical metric for understanding teenage behavior and well-being. A national health study investigates the amount of time teenagers spend on screens (e.g., phones, computers, TVs) per day. In 2023, the daily screen time of teenagers in a large sample was normally distributed with a mean of 6 hours and a standard deviation of 1.5 hours.
Using this information, calculate the following probabilities or values for the daily screen time of teenagers:

1. The percentage of teenagers with daily screen time between 4.5 and 6 hours.

☐ %

2. The percentage of teenagers with daily screen time between 6 and 9 hours.

☐ %

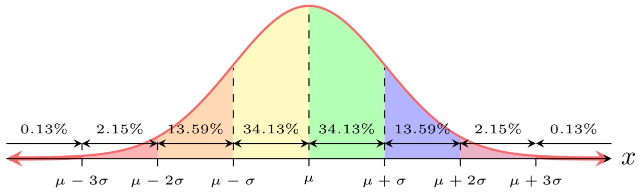
3. The percentage of teenagers with daily screen time less than 7.5 hours.

☐ %

4. In 2024, if there were 1200 teenagers in this sample, estimate the number of teenagers with daily screen time greater than 9 hours (round to the nearest integer).

☐ teenagers

For a normal distribution, the coverage probabilities are illustrated below:



B.2.4 FINDING PROBABILITIES USING GRAPHIC CALCULATOR

Ex 44: Suppose X represents the time (in minutes) taken to complete a task, and it follows a normal distribution with a mean of 40 minutes and a standard deviation of 10 minutes. Calculate the probability that the task is completed between 37 and 48 minutes. Round your answer to three decimal places.

$$P(37 \leqslant X \leqslant 48) \approx \square$$

Ex 45: Suppose X represents the annual rainfall (in millimeters) in a coastal city, and it follows a normal distribution with a mean of 1200 mm and a standard deviation of 150 mm. Calculate the probability that the annual rainfall exceeds 1350 mm. Round your answer to two decimal places.

$$P(X \geqslant 1350) \approx \square$$

Ex 46: Suppose X represents the Elo rating of a chess player, and it follows a normal distribution with a mean of 1500 and a standard deviation of 200. Calculate the probability that a player's rating exceeds 2000. Round your answer to three decimal places.

$$P(X \geqslant 2000) \approx \square$$

Ex 47: Suppose X represents the height (in centimeters) of adult women in Australia, and it follows a normal distribution with a mean of 165 cm and a standard deviation of 7 cm. Calculate the probability that a woman's height is less than or equal to 160 cm. Round your answer to three decimal places.

$$P(X \leqslant 160) \approx \square$$

B.2.5 BUSTING BRAGS AND CLAIMS WITH NORMAL CURVES

Ex 48: Suppose X represents the scores (in points) of students in a math class evaluation, and it follows a normal distribution with a mean of 65 points and a standard deviation of 10 points. Hugo receives a score of 75 points and claims, "I am in the top 2% of students in this class."
Are you agree with Hugo ? Explains your answer.

Ex 49: Suppose X represents the daily water consumption (in liters) of households in a small town, and it follows a normal distribution with a mean of 200 liters and a standard deviation of 30 liters. Maria measures her household's consumption as 260 liters and claims, "We are in the top 2% of households in this town."
Are you agree with Maria ? Explains your answer.

Ex 50: Suppose X represents the height (in centimeters) of boys in a school, and it follows a normal distribution with a mean of 175 cm and a standard deviation of 8 cm. The school states, "95% of boys can pass under a door of height 190 cm."
Are you agree with this statement ? Explains your answer.

Ex 51: Suppose X represents the high scores (in points) of players in a new battle royale video game, and it follows a normal distribution with a mean of 500 points and a standard deviation of 50 points. Liam gets a high score of 600 points and brags, "I'm in the top 5% of all players!"
Are you agree with Liam ? Explains your answer.

B.3 QUANTILE

B.3.1 SETTING THE THRESHOLD WITH PERCENTILES

Ex 52: Suppose X represents the time (in minutes) taken to complete a task, and it follows a normal distribution with a mean of 40 minutes and a standard deviation of 10 minutes. The teacher fixes the duration of the exam such that 95% of students have finished. Find this time (i.e., the 95th percentile). Round your answer to one decimal place.

$x \approx$ ☐

Ex 53: Suppose X represents the delivery time (in minutes) of pizzas from a local shop, and it follows a normal distribution with a mean of 25 minutes and a standard deviation of 5 minutes. The shop guarantees a delivery deadline such that 90% of orders are delivered before this time. Find this time (i.e., the 90th percentile). Round your answer to one decimal place.

$x \approx$ ☐

Ex 54: Suppose X represents the height (in centimeters) of men, and it follows a normal distribution with a mean of 175.3 cm and a standard deviation of 7.1 cm. A builder wants to design a door height such that at least 95% of men can walk through without ducking. Find this height (i.e., the 95th percentile). Round your answer to one decimal place.

$x \approx$ ☐

Ex 55: Suppose X represents the battery life (in hours) of a new smartphone model, and it follows a normal distribution with a mean of 12 hours and a standard deviation of 2 hours. The manufacturer sets a warranty replacement time such that 80% of phones last at least this long before needing a recharge. Find this time (i.e., the 20th percentile, since it's the lower tail). Round your answer to one decimal place.

$x \approx$ ☐

Ex 56: Suppose X represents the noise level (in decibels) of a crowd at a school concert, and it follows a normal distribution with a mean of 85 decibels and a standard deviation of 15 decibels. The sound engineer sets a microphone threshold such that 60% of the time, the noise is below this level. Find this noise level (i.e., the 60th percentile). Round your answer to one decimal place.

$x \approx$ ☐

Ex 57: Suppose X represents the weight (in kilograms) of backpacks carried by students, and it follows a normal distribution with a mean of 8 kg and a standard deviation of 1.5 kg. The school sets a minimum weight limit for a strength training program such that 95% of students carry at least this weight. Find this weight (i.e., the 5th percentile, since it's the lower tail). Round your answer to one decimal place.

$x \approx \boxed{}$