COUPLED DIFFERENTIAL EQUATIONS

A DEFINITIONS AND EQUILIBRIUM

A.1 CHECKING A SOLUTION

Ex 1: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 2x + y \end{cases}$$

Show that the functions defined by $x(t) = e^{3t}$ and $y(t) = e^{3t}$ form a valid solution to this system.

Answer: To verify the solution, we calculate the derivatives and substitute into the system.

- **Derivatives:**

$$\frac{d}{dt}(x(t)) = \frac{d}{dt}(e^{3t}) = 3e^{3t}$$
$$\frac{d}{dt}(y(t)) = \frac{d}{dt}(e^{3t}) = 3e^{3t}$$

- **Verification of the first equation ($\frac{d}{dt}(x(t)) = x(t) + 2y(t)$):**

$$\text{LHS} = 3e^{3t}$$
$$\text{RHS} = e^{3t} + 2(e^{3t}) = 3e^{3t}$$

Since LHS = RHS, the first equation is satisfied.

- **Verification of the second equation ($\frac{d}{dt}(y(t)) = 2x(t) + y(t)$):**

$$\text{LHS} = 3e^{3t}$$
$$\text{RHS} = 2(e^{3t}) + e^{3t} = 3e^{3t}$$

Since LHS = RHS, the second equation is satisfied.

Conclusion: The pair $(x(t), y(t))$ is indeed a solution to the system.

Ex 2: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}$$

Show that the functions defined by $x(t) = \sin(t)$ and $y(t) = \cos(t)$ form a valid solution to this system.

Answer: To verify the solution, we calculate the derivatives and substitute into the system.

- **Derivatives:**

$$\frac{d}{dt}(x(t)) = \frac{d}{dt}(\sin(t)) = \cos(t)$$
$$\frac{d}{dt}(y(t)) = \frac{d}{dt}(\cos(t)) = -\sin(t)$$

- **Verification of the first equation ($\frac{d}{dt}(x(t)) = y(t)$):**

$$\text{LHS} = \cos(t)$$
$$\text{RHS} = \cos(t)$$

Since LHS = RHS, the first equation is satisfied.

- **Verification of the second equation ($\frac{d}{dt}(y(t)) = -x(t)$):**

$$\text{LHS} = -\sin(t)$$
$$\text{RHS} = -\sin(t)$$

Since LHS = RHS, the second equation is satisfied.

Conclusion: The pair $(x(t), y(t))$ is indeed a solution to the system.

Ex 3: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = y^2 \\ \dfrac{dy}{dt} = x \end{cases}$$

Show that the functions defined by $x(t) = -\dfrac{12}{t^3}$ and $y(t) = \dfrac{6}{t^2}$ form a valid solution to this system for $t \neq 0$.

Answer: To verify the solution, we calculate the derivatives and substitute into the system. Recall that $\frac{d}{dt}(t^n) = nt^{n-1}$.

- **Derivatives:**

$$\frac{d}{dt}(x(t)) = \frac{d}{dt}(-12t^{-3}) = 36t^{-4} = \frac{36}{t^4}$$
$$\frac{d}{dt}(y(t)) = \frac{d}{dt}(6t^{-2}) = -12t^{-3} = -\frac{12}{t^3}$$

- **Verification of the first equation ($\frac{d}{dt}(x(t)) = (y(t))^2$):**

$$\text{LHS} = \frac{36}{t^4}$$
$$\text{RHS} = \left(\frac{6}{t^2}\right)^2 = \frac{36}{t^4}$$

Since LHS = RHS, the first equation is satisfied.

- **Verification of the second equation ($\frac{d}{dt}(y(t)) = x(t)$):**

$$\text{LHS} = -\frac{12}{t^3}$$
$$\text{RHS} = -\frac{12}{t^3}$$

Since LHS = RHS, the second equation is satisfied.

Conclusion: The pair $(x(t), y(t))$ is indeed a solution to the system.

A.2 IDENTIFYING COUPLED SYSTEMS

MCQ 4: Which of the following systems of differential equations is **coupled**?

- ☐ $\dfrac{dx}{dt} = 3x$
- ☒ $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = x - y \end{cases}$
- ☐ $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \end{cases}$

Answer: A system is **coupled** if the equations depend on each other (you cannot solve one without the other).

- The first option is a single differential equation, not a system.
- The second option $\begin{cases}\dot{x} = x + y \\ \dot{y} = x - y\end{cases}$ is **coupled** because the rate of change of x depends on y, and vice versa.
- The third option $\begin{cases}\dot{x} = x \\ \dot{y} = y\end{cases}$ is **uncoupled** (or decoupled) because $\dot{x}$ depends only on x and $\dot{y}$ depends only on y. They can be solved separately.

MCQ 5: Which of the following systems of differential equations is **coupled**?

☐ $\begin{cases}\dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 5y\end{cases}$

☐ $\dfrac{dy}{dt} = 3y^2 - t$

☒ $\begin{cases}\dfrac{dx}{dt} = 3x - 2y \\ \dfrac{dy}{dt} = x + 4y\end{cases}$

Answer:

- The first option $\begin{cases}\dot{x} = 2x \\ \dot{y} = 5y\end{cases}$ is **uncoupled** because the rate of change of x depends only on x, and y only on y. They can be solved independently ($x = Ae^{2t}, y = Be^{5t}$).
- The second option is a single differential equation, not a system.
- The third option $\begin{cases}\dot{x} = 3x - 2y \\ \dot{y} = x + 4y\end{cases}$ is **coupled** because $\dot{x}$ depends on y and $\dot{y}$ depends on x. They must be solved together.

A.3 DETERMINING THE EQUILIBRIUM POINT(S)

Ex 6: Consider the following system of coupled differential equations:

$$\begin{cases}\dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = x + 2y\end{cases}$$

Determine the equilibrium point(s) of the system.

Answer: Equilibrium when $\dot{x} = 0$ and $\dot{y} = 0$:

$$\begin{cases}2x - y = 0 \\ x + 2y = 0\end{cases}$$
$$\Leftrightarrow \begin{cases}2x = y \\ x + 2y = 0\end{cases}$$
$$\Leftrightarrow \begin{cases}y = 2x \\ x + 2(2x) = 0 \quad \text{(substitution } y = 2x\text{)}\end{cases}$$
$$\Leftrightarrow \begin{cases}y = 2x \\ 5x = 0\end{cases}$$
$$\Leftrightarrow \begin{cases}y = 0 \quad \text{(substitution } x = 0\text{)} \\ x = 0\end{cases}$$

The only equilibrium point is $(0, 0)$.

Ex 7: Consider the following system of coupled differential equations:

$$\begin{cases}\dfrac{dx}{dt} = x - 1 \\ \dfrac{dy}{dt} = x + y\end{cases}$$

Determine the equilibrium point(s) of the system.

Answer: Equilibrium when $\dot{x} = 0$ and $\dot{y} = 0$:

$$\begin{cases}x - 1 = 0 \\ x + y = 0\end{cases}$$
$$\Longleftrightarrow \begin{cases}x = 1 \\ 1 + y = 0 \quad \text{(substitution } x = 1\text{)}\end{cases}$$
$$\Longleftrightarrow \begin{cases}x = 1 \\ y = -1\end{cases}$$

The only equilibrium point is $(1, -1)$.

Ex 8: Consider the following system of coupled differential equations:

$$\begin{cases}\dfrac{dx}{dt} = x^2 - 1 \\ \dfrac{dy}{dt} = x + y\end{cases}$$

Determine the equilibrium point(s) of the system.

Answer: Equilibrium occurs when the derivatives are zero:

$$\begin{cases}x^2 - 1 = 0 \\ x + y = 0\end{cases}$$
$$\Longleftrightarrow \begin{cases}x^2 = 1 \\ y = -x\end{cases}$$
$$\Longleftrightarrow \begin{cases}x = 1 \text{ or } x = -1 \\ y = -x\end{cases}$$

We find the corresponding y values for each x:

- If $x = 1$, then $y = -1$.
- If $x = -1$, then $y = -(-1) = 1$.

The equilibrium points are $(1, -1)$ and $(-1, 1)$.

B PHASE PORTRAIT

B.1 CALCULATING VELOCITY VECTORS

Ex 9: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(1, 1)$.

$$\begin{pmatrix} \boxed{1} \\ \boxed{3} \end{pmatrix}$$

2. Find the velocity vector at the point $(-1, 1)$.

$$\begin{pmatrix} \boxed{-3} \\ \boxed{1} \end{pmatrix}$$

Answer:

1. At $(1, 1)$: $\dot{x} = 2(1) - 1 = 1$, $\dot{y} = 1 + 2(1) = 3$. Vector: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
2. At $(-1, 1)$: $\dot{x} = 2(-1) - 1 = -3$, $\dot{y} = -1 + 2(1) = 1$. Vector: $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

Ex 10: Consider the following system of non-linear coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = -x^2 + y \\ \dfrac{dy}{dt} = -(x - y)^2 \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(2, 5)$.

$$\begin{pmatrix} \boxed{1} \\ \boxed{-9} \end{pmatrix}$$

2. Find the velocity vector at the point $(1, 0)$.

$$\begin{pmatrix} \boxed{-1} \\ \boxed{-1} \end{pmatrix}$$

Answer:

1. At $(2, 5)$:
$$\dot{x} = -(2)^2 + 5 = -4 + 5 = 1$$
$$\dot{y} = -(2 - 5)^2 = -(-3)^2 = -(9) = -9$$
Vector: $\begin{pmatrix} 1 \\ -9 \end{pmatrix}$.
2. At $(1, 0)$:
$$\dot{x} = -(1)^2 + 0 = -1$$
$$\dot{y} = -(1 - 0)^2 = -(1)^2 = -1$$
Vector: $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

Ex 11: Consider the following system involving products of variables:

$$\begin{cases} \dfrac{dx}{dt} = xy + 1 \\ \dfrac{dy}{dt} = y^2 - 3x \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(1, 2)$.

$$\begin{pmatrix} \boxed{3} \\ \boxed{1} \end{pmatrix}$$

2. Find the velocity vector at the point $(2, -1)$.

$$\begin{pmatrix} \boxed{-1} \\ \boxed{-5} \end{pmatrix}$$

Answer:

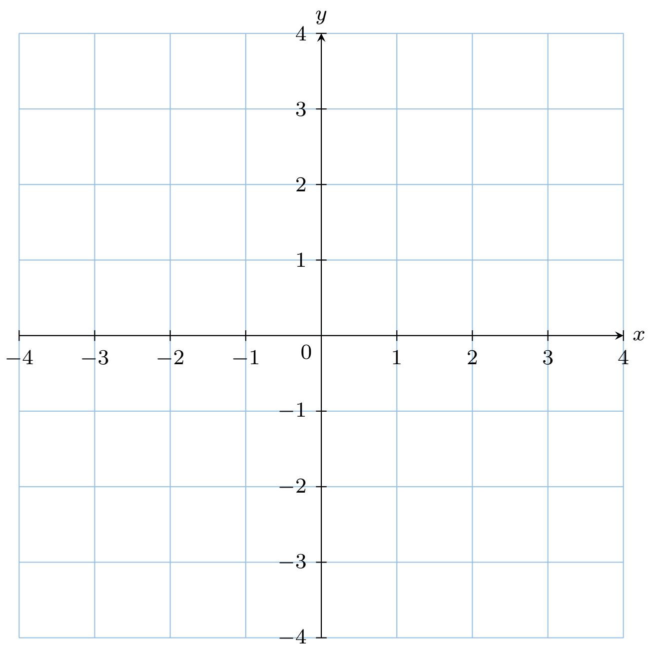
1. At $(1, 2)$:
$$\dot{x} = (1)(2) + 1 = 3$$
$$\dot{y} = (2)^2 - 3(1) = 4 - 3 = 1$$
Vector: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
2. At $(2, -1)$:
$$\dot{x} = (2)(-1) + 1 = -2 + 1 = -1$$
$$\dot{y} = (-1)^2 - 3(2) = 1 - 6 = -5$$
Vector: $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$.

B.2 SKETCHING PHASE PORTRAITS

Ex 12: Consider the following system of coupled differential equations:

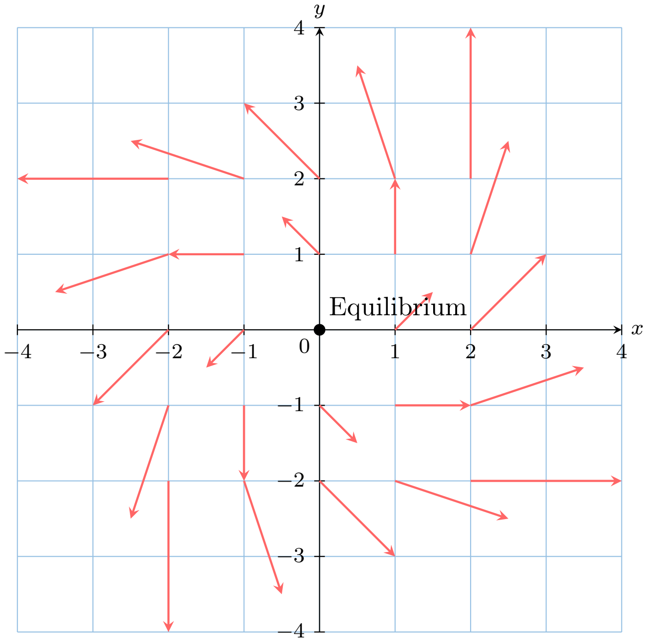
$$\begin{cases} \dfrac{dx}{dt} = \frac{x-y}{2} \\ \dfrac{dy}{dt} = \frac{x+y}{2} \end{cases}$$

Sketch the phase portrait for the system by drawing the velocity vector at each grid point (x, y) where $x, y \in \{-2, -1, 0, 1, 2\}$.



Answer: To draw the phase portrait, we calculate the vector $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\frac{x-y}{2}\\ \frac{x+y}{2}\end{pmatrix}$ for each integer point. For example:

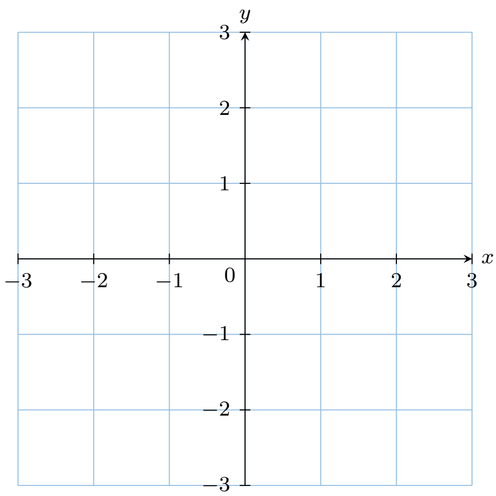
- At $(1, 0)$: vector is $\begin{pmatrix}0.5\\ 0.5\end{pmatrix}$.
- At $(0, 1)$: vector is $\begin{pmatrix}-0.5\\ 0.5\end{pmatrix}$.



Ex 13: Consider the following system of coupled differential equations:

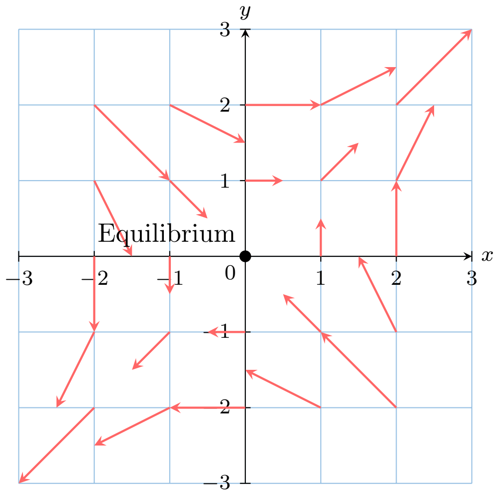
$$\begin{cases}\dfrac{dx}{dt} = y\\ \dfrac{dy}{dt} = x\end{cases}$$

Sketch the phase portrait for the system by drawing the velocity vector at each grid point (x, y) where $x, y \in \{-2, -1, 0, 1, 2\}$. The vectors should be scaled by a factor of 0.5 for clarity.



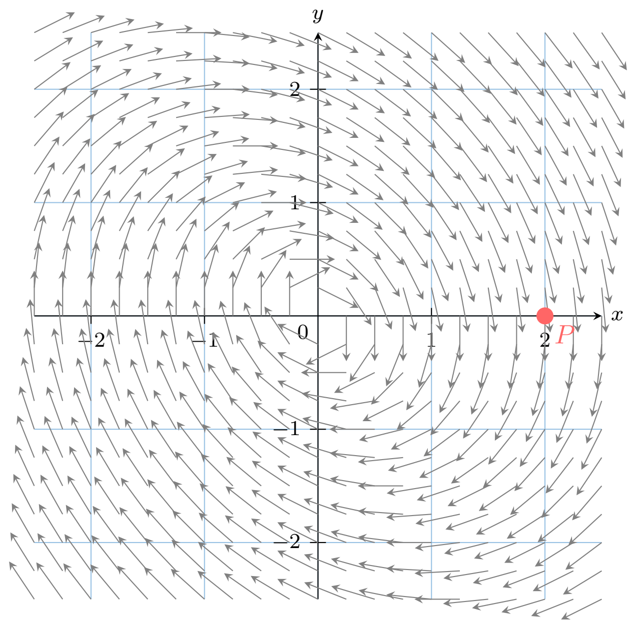
Answer: To draw the phase portrait, we calculate the vector $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}y\\ x\end{pmatrix}$ for each integer point. For example:

- At $(1, 0)$: vector is $\begin{pmatrix}0\\ 1\end{pmatrix}$ (vertical up).
- At $(0, 1)$: vector is $\begin{pmatrix}1\\ 0\end{pmatrix}$ (horizontal right).
- At $(1, 1)$: vector is $\begin{pmatrix}1\\ 1\end{pmatrix}$ (diagonal up-right).

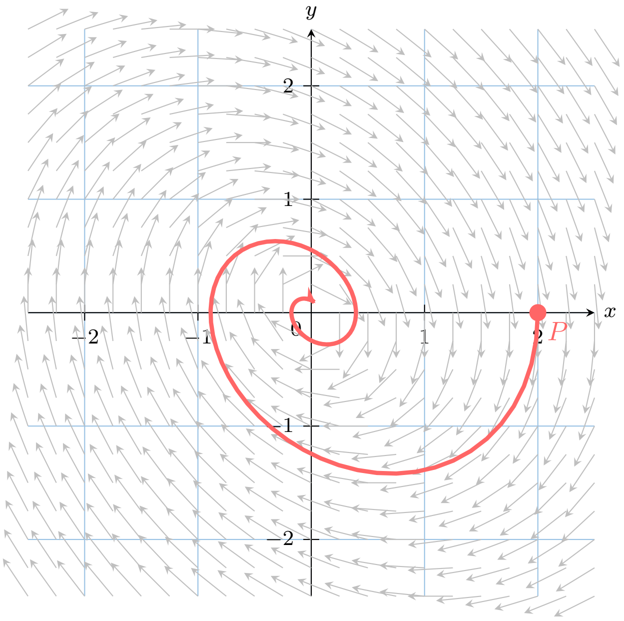


B.3 SKETCHING TRAJECTORIES FROM PHASE PORTRAITS

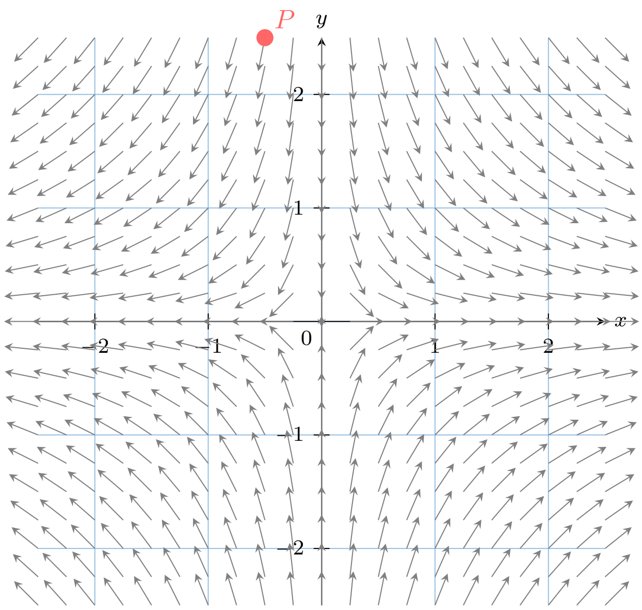
Ex 14: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(2, 0)$, sketch the **trajectory** of the system.
Follow the flow of the arrows carefully.



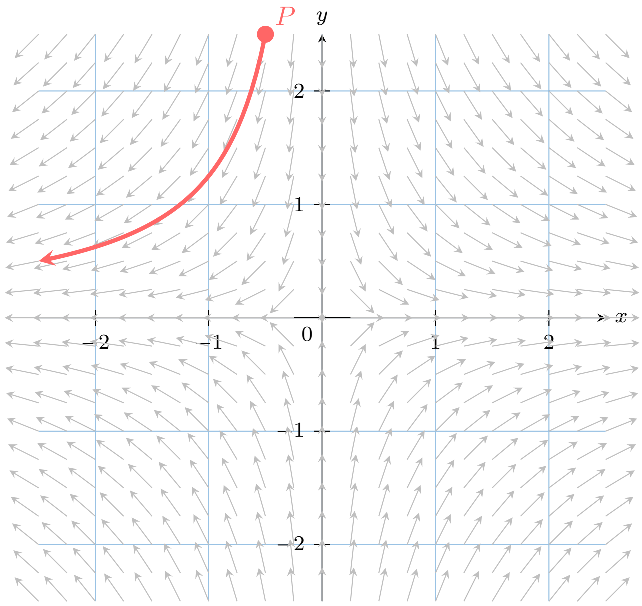
Answer: The trajectory starts at $P(2, 0)$. Following the arrows, it moves downwards, curves to the left, crosses the negative y-axis, and spirals inwards towards the center.



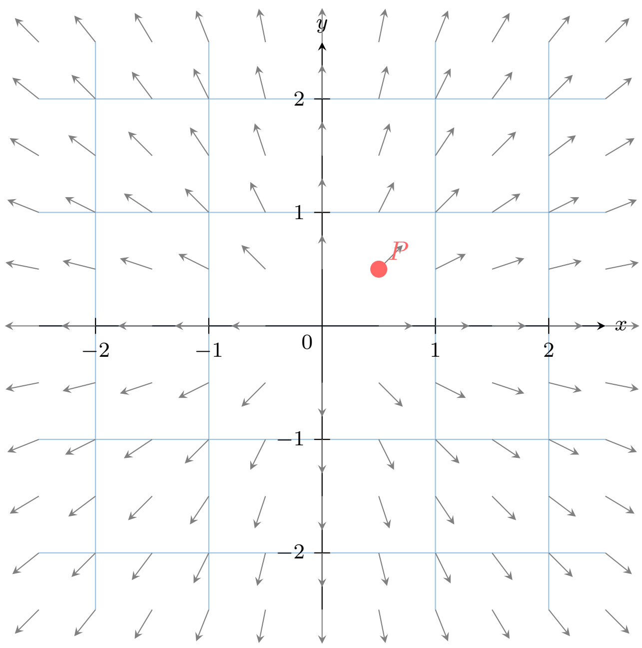
Ex 15: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(-0.5, 2.5)$, sketch the **trajectory** of the system. Follow the flow of the arrows carefully.



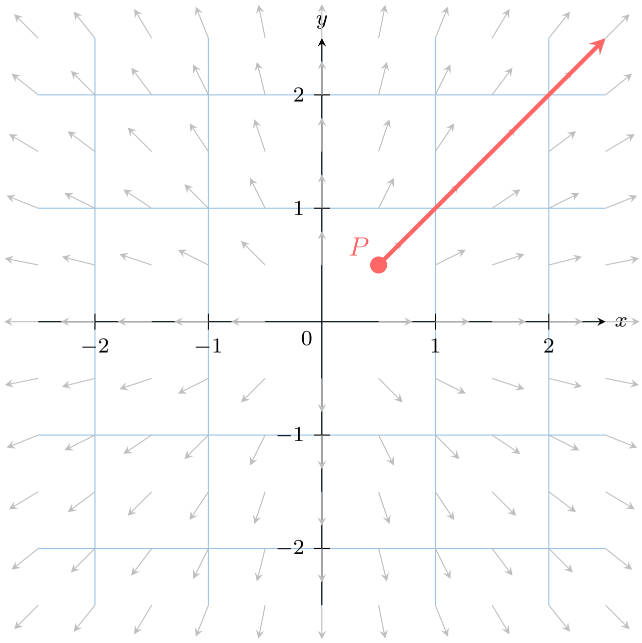
Answer: The trajectory starts at $P(-0.5, 2.5)$.



Ex 16: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(0.5, 0.5)$, sketch the **trajectory** of the system. Follow the flow of the arrows carefully.



Answer:



C COUPLED LINEAR DIFFERENTIAL EQUATIONS

C.1 WRITING COUPLED SYSTEMS IN MATRIX FORM

Ex 17: Consider the system of coupled linear differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Answer:

$$\begin{cases} \dot{x} = 2x - y \\ \dot{y} = 3x + 2y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x - y \\ 3x + 2y \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$.

Ex 18: Consider the system of coupled linear differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 4x + 7y \\ \dfrac{dy}{dt} = -2x + 5y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Answer:

$$\begin{cases} \dot{x} = 4x + 7y \\ \dot{y} = -2x + 5y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4x + 7y \\ -2x + 5y \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ -2 & 5 \end{pmatrix}$.

Ex 19: Consider the system of coupled linear differential equations:

$$\begin{cases} \dfrac{dx}{dt} = -x + 3y \\ \dfrac{dy}{dt} = 6x \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Answer: The second equation can be written as $\dot{y} = 6x + 0y$.

$$\begin{cases} \dot{x} = -1x + 3y \\ \dot{y} = 6x + 0y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1x + 3y \\ 6x + 0y \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 6 & 0 \end{pmatrix}$.

Ex 20: Consider the system of coupled linear differential equations:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Answer: The first equation is $\dot{x} = 0x + 1y$.

$$\begin{cases} \dot{x} = 0x + 1y \\ \dot{y} = -1x - 1y \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0x + 1y \\ -1x - 1y \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

C.2 SOLVING COUPLED LINEAR DIFFERENTIAL EQUATIONS

Ex 21: Consider the system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = x + y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.
4. Hence, write the general solution for the system.

Answer:

1. **Matrix Form:**

$$\begin{aligned} &\begin{cases} \dot{x} = 4x - 2y \\ \dot{y} = x + y \end{cases} \\ \Leftrightarrow &\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4x - 2y \\ x + y \end{pmatrix} \\ \Leftrightarrow &\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

   So $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

2. **Find eigenvalues:**

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ (4-\lambda)(1-\lambda) - (-2)(1) &= 0 \\ 4 - 5\lambda + \lambda^2 + 2 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ (\lambda - 3)(\lambda - 2) &= 0 \end{aligned}$$

   The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

3. **Find eigenvector for $\lambda_1 = 3$:**

$$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 3\begin{pmatrix}x\\ y\end{pmatrix} \implies \begin{pmatrix}1 & -2\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$x - 2y = 0 \implies x = 2y$. Let $y = 1$, then $x = 2$. So $\mathbf{v}_1 = \begin{pmatrix}2\\ 1\end{pmatrix}$.

Find eigenvector for $\lambda_2 = 2$:

$$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 2\begin{pmatrix}x\\ y\end{pmatrix} \implies \begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$x - y = 0 \implies x = y$. Let $y = 1$, then $x = 1$. So $\mathbf{v}_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$.

4. **General Solution:**

$$\mathbf{x}(t) = Ae^{3t}\begin{pmatrix}2\\ 1\end{pmatrix} + Be^{2t}\begin{pmatrix}1\\ 1\end{pmatrix}$$

Or written separately:

$$\begin{cases} x(t) = 2Ae^{3t} + Be^{2t} \\ y(t) = Ae^{3t} + Be^{2t} \end{cases}$$

Ex 22: Consider the system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 5x + 4y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.
4. Hence, write the general solution for the system.

Answer:

1. **Matrix Form:**

$$\begin{cases} \dot{x} = 5x + 4y \\ \dot{y} = x + 2y \end{cases}$$
$$\Leftrightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}5x + 4y\\ x + 2y\end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}$.

2. **Find eigenvalues:**

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ (5-\lambda)(2-\lambda) - (4)(1) &= 0 \\ 10 - 7\lambda + \lambda^2 - 4 &= 0 \\ \lambda^2 - 7\lambda + 6 &= 0 \\ (\lambda - 6)(\lambda - 1) &= 0 \end{aligned}$$

The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 1$.

3. **Find eigenvector for $\lambda_1 = 6$:**

$$\begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 6\begin{pmatrix}x\\ y\end{pmatrix} \implies \begin{pmatrix}-1 & 4\\ 1 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$x - 4y = 0 \implies x = 4y$. Let $y = 1$, then $x = 4$. So $\mathbf{v}_1 = \begin{pmatrix}4\\ 1\end{pmatrix}$.

Find eigenvector for $\lambda_2 = 1$:

$$\begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 1\begin{pmatrix}x\\ y\end{pmatrix} \implies \begin{pmatrix}4 & 4\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$x + y = 0 \implies x = -y$. Let $y = -1$, then $x = 1$. So $\mathbf{v}_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$.

4. **General Solution:**

$$\mathbf{x}(t) = Ae^{6t}\begin{pmatrix}4\\ 1\end{pmatrix} + Be^{t}\begin{pmatrix}1\\ -1\end{pmatrix}$$

Or written separately:

$$\begin{cases} x(t) = 4Ae^{6t} + Be^{t} \\ y(t) = Ae^{6t} - Be^{t} \end{cases}$$

Ex 23: Consider the system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 4x + 2y \\ \dfrac{dy}{dt} = 3x - y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.
4. Hence, write the general solution for the system.

Answer:

1. **Matrix Form:**

$$\begin{cases} \dot{x} = 4x + 2y \\ \dot{y} = 3x - y \end{cases}$$
$$\Leftrightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}4x + 2y\\ 3x - y\end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

So $\mathbf{A} = \begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}$.

2. **Find eigenvalues:**

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ (4-\lambda)(-1-\lambda) - (2)(3) &= 0 \\ -4 - 3\lambda + \lambda^2 - 6 &= 0 \\ \lambda^2 - 3\lambda - 10 &= 0 \\ (\lambda - 5)(\lambda + 2) &= 0 \end{aligned}$$

The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$.

3. **Find eigenvector for $\lambda_1 = 5$:**

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x + 2y = 0 \implies x = 2y$. Let $y = 1$, then $x = 2$. So $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Find eigenvector for $\lambda_2 = -2$:

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x + y = 0 \implies y = -3x$. Let $x = 1$, then $y = -3$. So $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

4. **General Solution:**

$$\mathbf{x}(t) = C_1 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Or written separately:

$$\begin{cases} x(t) = 2C_1 e^{5t} + C_2 e^{-2t} \\ y(t) = C_1 e^{5t} - 3C_2 e^{-2t} \end{cases}$$

D SECOND ORDER DIFFERENTIAL EQUATIONS

D.1 CONVERTING SECOND-ORDER EQUATIONS TO SYSTEMS

Ex 24: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Answer:

1. Let $y = \frac{dx}{dt}$. Then:

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

From the original equation, we rearrange to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -2x - 3\frac{dx}{dt}$$

Substituting y into this expression:

$$\frac{dy}{dt} = -2x - 3y$$

The system of coupled equations is:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -2x - 3y \end{cases}$$

2.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 3y \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -2x - 3y \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Ex 25: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Answer:

1. Let $y = \frac{dx}{dt}$. Then:

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

From the original equation, we rearrange to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -6x - 5\frac{dx}{dt}$$

Substituting y into this expression:

$$\frac{dy}{dt} = -6x - 5y$$

The system of coupled equations is:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -6x - 5y \end{cases}$$

2.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -6x - 5y \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -6x - 5y \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Ex 26: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Answer:

1. Let $y = \frac{dx}{dt}$. Then:

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

From the original equation, we rearrange to isolate the second derivative:

$$\frac{d^2x}{dt^2} = -3x + 4\frac{dx}{dt}$$

Substituting y into this expression:

$$\frac{dy}{dt} = -3x + 4y$$

The system of coupled equations is:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -3x + 4y \end{cases}$$

2.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -3x + 4y \end{cases}$$
$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -3x + 4y \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Ex 27: When an object with mass m falls under gravity, its displacement x is governed by the equation of motion:

$$m\ddot{x} = -mg - k(\dot{x})^2$$

where $g \approx 9.8$ is the acceleration due to gravity and k is a positive constant representing air resistance.
Use the substitution $v = \dot{x}$ to write this as a coupled system of differential equations involving x and v.

Answer: Let $v = \dot{x}$. Then the derivative of velocity is acceleration: $\dot{v} = \ddot{x}$.
Substituting these into the original equation:

$$m\dot{v} = -mg - k(v)^2$$
$$\dot{v} = -g - \frac{k}{m}v^2$$

The system of coupled equations is:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -g - \frac{k}{m}v^2 \end{cases}$$

(Note: This is a non-linear system because of the v^2 term, so it cannot be written in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$).

D.2 MODELLING PHYSICAL SYSTEMS WITH SECOND-ORDER EQUATIONS

Ex 28: The temperature x of a drink t seconds after ice cubes are added to it, satisfies the differential equation

$$50\frac{d^2x}{dt^2} + \frac{dx}{dt} = 0$$

1. Use the substitution $y = \frac{dx}{dt}$ to write this as a coupled system of first order differential equations.
2. The equation for $\frac{dy}{dt}$ is separable and independent of x. Solve this equation for $y(t)$.
3. Hence find a general solution for $x(t)$.

Answer:

1. **Coupled System:**
Let $y = \frac{dx}{dt}$. Then $\frac{dy}{dt} = \frac{d^2x}{dt^2}$. Substituting into the original equation:
$$50\frac{dy}{dt} + y = 0$$
Rearranging gives:
$$\frac{dy}{dt} = -\frac{1}{50}y$$
The coupled system is therefore:
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{1}{50}y \end{cases}$$

2. **Solve for $y(t)$:**
The second equation is separable:
$$\frac{dy}{dt} = -\frac{1}{50}y \implies \frac{dy}{y} = -\frac{1}{50}\,dt \quad (y \neq 0)$$
Integrating both sides:
$$\int \frac{dy}{y} = -\frac{1}{50}\int dt$$
$$\ln|y| = -\frac{1}{50}t + C_1$$
$$y = e^{C_1} \cdot e^{-t/50} = Ae^{-t/50}$$
where $A = e^{C_1}$ is an arbitrary constant (which can be positive, negative, or zero, though $y = 0$ is the trivial case already included when $A = 0$).

3. **Find $x(t)$:**
Since $\frac{dx}{dt} = y = Ae^{-t/50}$,
$$x(t) = \int Ae^{-t/50}\,dt = A\cdot(-50)e^{-t/50} + B = -50Ae^{-t/50} + B$$
where B is another arbitrary constant. This is the general solution:
$$x(t) = B - 50Ae^{-t/50}$$
(Alternatively, rewriting the constants: $x(t) = C_1 + C_2e^{-t/50}$, where $C_1 = B$ and $C_2 = -50A$.)

Ex 29: The charge $q(t)$ (in coulombs) on the capacitor in a series RLC circuit satisfies the differential equation

$$20\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 0,$$

where t is time in seconds.

1. Use the substitution $i = \frac{dq}{dt}$ (where i represents the current in the circuit) to rewrite this equation as a coupled system of first-order differential equations.
2. The equation for $\frac{di}{dt}$ is separable and independent of q. Solve this equation to find $i(t)$.
3. Hence obtain the general solution for $q(t)$.

Answer:

1. **Coupled system**
Let $i = \frac{dq}{dt}$. Then $\frac{di}{dt} = \frac{d^2q}{dt^2}$. Substituting into the original equation:
$$20\frac{di}{dt} + 10i + 100q = 0.$$
Rearranging gives
$$\frac{di}{dt} = -\frac{10}{20}i - \frac{100}{20}q = -\frac{1}{2}i - 5q.$$
The coupled system is therefore
$$\begin{cases} \frac{dq}{dt} = i \\ \frac{di}{dt} = -5q - \frac{1}{2}i \end{cases}$$

2. **Solution for $i(t)$**
The second equation does not involve q, and is separable:
$$\frac{di}{dt} = -\frac{1}{2}i \implies \frac{di}{i} = -\frac{1}{2}\,dt \qquad (i \neq 0).$$
Integrating both sides:
$$\ln|i| = -\frac{1}{2}t + C_1,$$
so
$$i(t) = Ae^{-t/2},$$
where $A = \pm e^{C_1}$ is an arbitrary constant (the case $A = 0$ is included).

3. **General solution for $q(t)$**
Now $\frac{dq}{dt} = i(t) = Ae^{-t/2}$, so
$$q(t) = \int Ae^{-t/2}\,dt = A\cdot(-2)e^{-t/2} + B = -2Ae^{-t/2} + B,$$
where B is an arbitrary constant.
Thus the general solution is
$$q(t) = B - 2Ae^{-t/2}.$$
Equivalently, redefining the constants,
$$q(t) = C_1 + C_2e^{-t/2},$$
where $C_1 = B$ and $C_2 = -2A$.

E EULER'S METHOD FOR COUPLED SYSTEMS

E.1 APPLYING EULER'S METHOD STEP BY STEP

Ex 30: Consider the system
$$\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = x + y \end{cases}$$
with initial conditions $x(0) = 1$, $y(0) = 0$. Use Euler's method with step size $h = 0.2$ to approximate $x(0.4)$ and $y(0.4)$ after two steps.

Answer:

1. **Step 0:** $t_0 = 0$, $x_0 = 1$, $y_0 = 0$.
2. **Step 1 (to $t_1 = 0.2$):**
$$\left.\frac{dx}{dt}\right|_{(1,0)} = 1 - 0 = 1$$
$$\left.\frac{dy}{dt}\right|_{(1,0)} = 1 + 0 = 1$$
$$x_1 = x_0 + h\cdot 1 = 1 + 0.2\cdot 1 = 1.2$$
$$y_1 = y_0 + h\cdot 1 = 0 + 0.2\cdot 1 = 0.2$$
3. **Step 2 (to $t_2 = 0.4$):**
$$\left.\frac{dx}{dt}\right|_{(1.2,0.2)} = 1.2 - 0.2 = 1.0$$
$$\left.\frac{dy}{dt}\right|_{(1.2,0.2)} = 1.2 + 0.2 = 1.4$$
$$x_2 = x_1 + h\cdot 1.0 = 1.2 + 0.2\cdot 1.0 = 1.4$$
$$y_2 = y_1 + h\cdot 1.4 = 0.2 + 0.2\cdot 1.4 = 0.48$$
After two steps, the approximation is $x(0.4) \approx 1.4$ and $y(0.4) \approx 0.48$.

Ex 31: Consider the system
$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = x - 3y \end{cases}$$
with initial conditions $x(0) = 2$, $y(0) = 1$. Use Euler's method with step size $h = 0.1$ to find approximations for $x(0.2)$ and $y(0.2)$ after two steps.

Answer:

1. **Step 0:** $t_0 = 0$, $x_0 = 2$, $y_0 = 1$.
2. **Step 1:**
$$\left.\frac{dx}{dt}\right|_{(2,1)} = -2(2) + 1 = -3$$
$$\left.\frac{dy}{dt}\right|_{(2,1)} = 2 - 3(1) = -1$$
$$x_1 = 2 + 0.1(-3) = 1.7$$
$$y_1 = 1 + 0.1(-1) = 0.9$$
3. **Step 2:**
$$\left.\frac{dx}{dt}\right|_{(1.7,0.9)} = -2(1.7) + 0.9 = -2.5$$
$$\left.\frac{dy}{dt}\right|_{(1.7,0.9)} = 1.7 - 3(0.9) = -1.0$$
$$x_2 = 1.7 + 0.1(-2.5) = 1.45$$
$$y_2 = 0.9 + 0.1(-1.0) = 0.8$$
After two steps, $x(0.2) \approx 1.45$ and $y(0.2) \approx 0.8$.

Ex 32: Consider the non-linear system
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x + y^2 \end{cases}$$

with initial conditions $x(0) = 0$, $y(0) = 1$. Use Euler's method with step size $h = 0.1$ to approximate $x(0.3)$ and $y(0.3)$ after three steps.

Answer:

1. **Step 0:** $t_0 = 0, x_0 = 0, y_0 = 1$.
2. **Step 1 (to $t_1 = 0.1$):**

$$\left.\frac{dx}{dt}\right|_{(0,1)} = 1$$
$$\left.\frac{dy}{dt}\right|_{(0,1)} = -0 + (1)^2 = 1$$

$$x_1 = 0 + 0.1(1) = 0.1$$
$$y_1 = 1 + 0.1(1) = 1.1$$

3. **Step 2 (to $t_2 = 0.2$):**

$$\left.\frac{dx}{dt}\right|_{(0.1,1.1)} = 1.1$$
$$\left.\frac{dy}{dt}\right|_{(0.1,1.1)} = -0.1 + (1.1)^2 = -0.1 + 1.21 = 1.11$$

$$x_2 = 0.1 + 0.1(1.1) = 0.21$$
$$y_2 = 1.1 + 0.1(1.11) = 1.211$$

4. **Step 3 (to $t_3 = 0.3$):**

$$\left.\frac{dx}{dt}\right|_{(0.21,1.211)} = 1.211$$
$$\left.\frac{dy}{dt}\right|_{(0.21,1.211)} = -0.21 + (1.211)^2 \approx -0.21 + 1.4665 = 1.2565$$

$$x_3 = 0.21 + 0.1(1.211) = 0.3311$$
$$y_3 = 1.211 + 0.1(1.2565) \approx 1.3367$$

So, $x(0.3) \approx 0.331$ and $y(0.3) \approx 1.337$.