DIFFERENTIAL CALCULUS

A DERIVATIVE

A.1 RATE OF CHANGE

A.1.1 FINDING RATE OF CHANGE

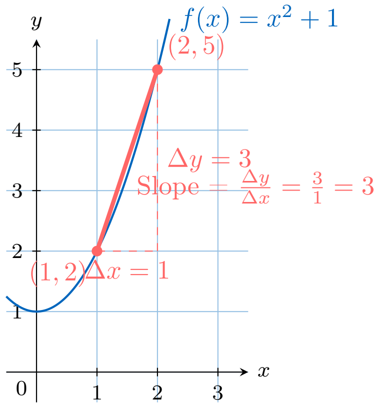
Ex 1: For the function $f(x) = x^2 + 1$, find the rate of change from $x = 1$ to $x = 2$.

$$\text{Rate of change} = \boxed{3}$$

Answer: We use the formula for the rate of change from $a = 1$ to $b = 2$.

$$\begin{aligned}\text{Rate of change} &= \frac{f(2) - f(1)}{2 - 1}\\ &= \frac{(2^2 + 1) - (1^2 + 1)}{1}\\ &= \frac{5 - 2}{1}\\ &= 3\end{aligned}$$

Geometrically, this value is the slope of the secant line connecting the points $(1, f(1))$ and $(2, f(2))$.



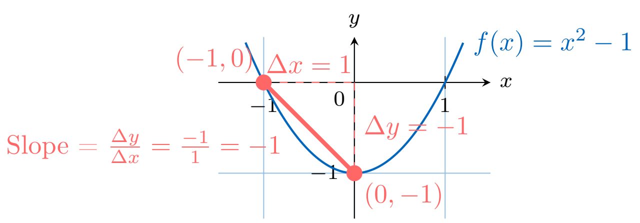
Ex 2: For the function $f(x) = (x - 1)(x + 1)$, find the rate of change from $x = -1$ to $x = 0$.

$$\text{Rate of change} = \boxed{-1}$$

Answer: First, we simplify the function: $f(x) = x^2 - 1$. We then use the formula for the rate of change from $a = -1$ to $b = 0$.

$$\begin{aligned}\text{Rate of change} &= \frac{f(0) - f(-1)}{0 - (-1)}\\ &= \frac{(0^2 - 1) - ((-1)^2 - 1)}{1}\\ &= \frac{(-1) - (1 - 1)}{1}\\ &= -1\end{aligned}$$

Geometrically, this value is the slope of the secant line connecting the points $(-1, f(-1))$ and $(0, f(0))$.



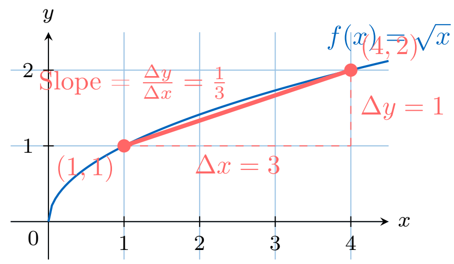
Ex 3: For the function $f(x) = \sqrt{x}$, find the rate of change from $x = 1$ to $x = 4$.

$$\text{Rate of change} = \boxed{1/3}$$

Answer: We use the formula for the rate of change from $a = 1$ to $b = 4$.

$$\begin{aligned}\text{Rate of change} &= \frac{f(4) - f(1)}{4 - 1}\\ &= \frac{\sqrt{4} - \sqrt{1}}{3}\\ &= \frac{2 - 1}{3}\\ &= \frac{1}{3}\end{aligned}$$

Geometrically, this value is the slope of the secant line connecting the points $(1, f(1))$ and $(4, f(4))$.



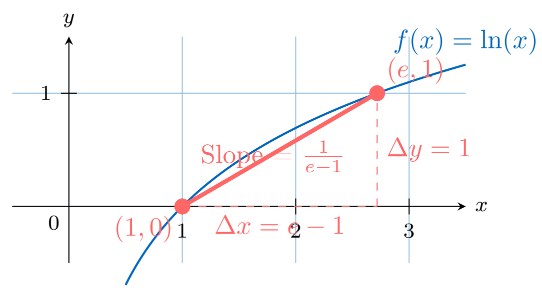
Ex 4: For the function $f(x) = \ln(x)$, find the rate of change from $x = 1$ to $x = e$.

$$\text{Rate of change} = \boxed{\frac{1}{e - 1}}$$

Answer: We use the formula for the rate of change from $a = 1$ to $b = e$.

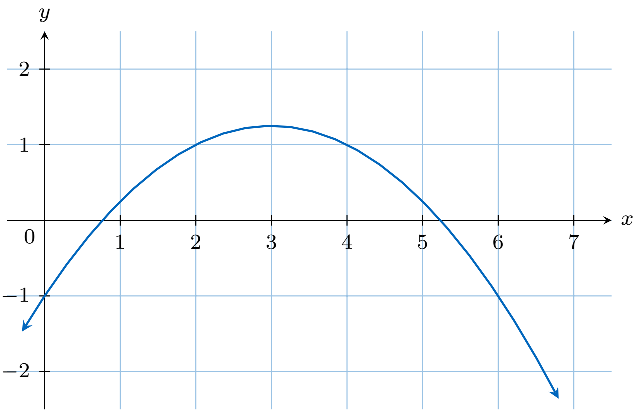
$$\begin{aligned}\text{Rate of change} &= \frac{f(e) - f(1)}{e - 1}\\ &= \frac{\ln(e) - \ln(1)}{e - 1}\\ &= \frac{1 - 0}{e - 1}\\ &= \frac{1}{e - 1}\end{aligned}$$

Geometrically, this value is the slope of the secant line connecting the points $(1, f(1))$ and $(e, f(e))$.



A.1.2 FINDING RATE OF CHANGE FROM A GRAPH

Ex 5: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 2$ to $x = 6$ graphically.

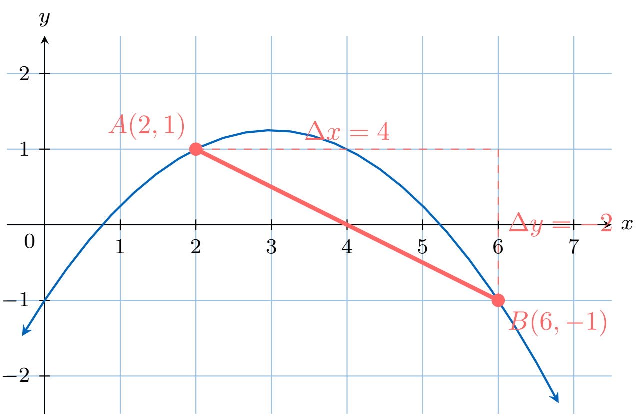
$$\text{Rate of change} = \boxed{-1/2}$$

Answer: The rate of change is the slope of the secant line connecting the points on the curve at $x = 2$ and $x = 6$.
From the graph, we can identify the coordinates of these two points:

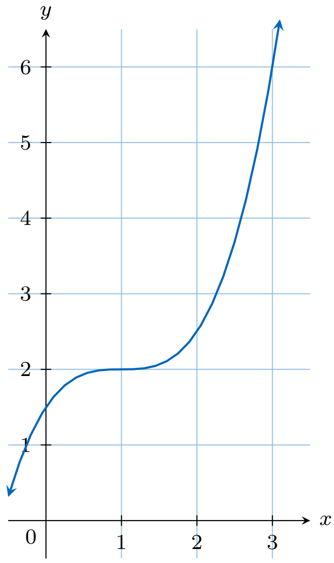
- At $x = 2$, the value of the function is $y = 1$. So we have the point $A(2, 1)$.
- At $x = 6$, the value of the function is $y = -1$. So we have the point $B(6, -1)$.

We can now calculate the slope of the secant line (AB).

$$\begin{aligned}\text{Rate of change} &= \text{Slope}\\ &= \frac{\Delta y}{\Delta x}\\ &= \frac{y_B - y_A}{x_B - x_A}\\ &= \frac{-1-1}{6-2}\\ &= \frac{-2}{4}\\ &= -\frac{1}{2}\end{aligned}$$



Ex 6: The graph of a function $y = f(x)$ is shown below.

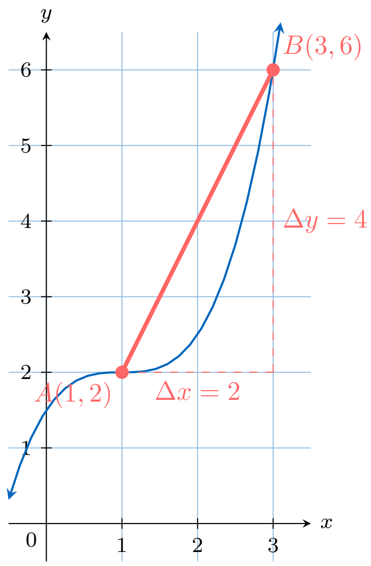


Find the rate of change of the function from $x = 1$ to $x = 3$ graphically.

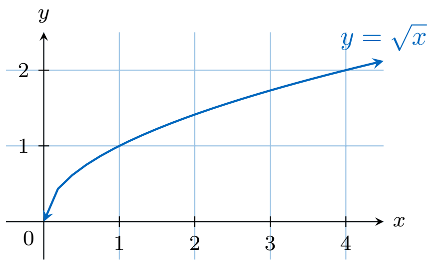
$$\text{Rate of change} = \boxed{2}$$

Answer: The rate of change is the slope of the secant line connecting the points on the curve at $x = 1$ and $x = 3$.
From the graph, we identify the coordinates: $A(1, 2)$ and $B(3, 6)$.

$$\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{6-2}{3-1} = \frac{4}{2} = 2$$



Ex 7: The graph of a function $y = f(x)$ is shown below.

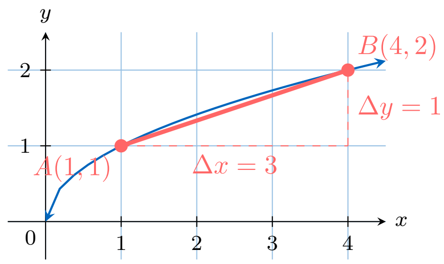


Find the rate of change of the function from $x = 1$ to $x = 4$ graphically.

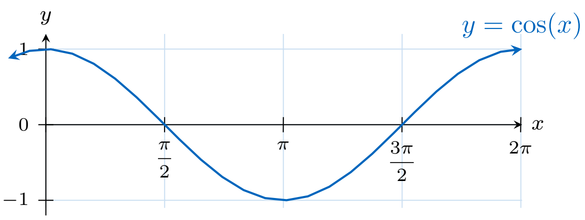
$$\text{Rate of change} = \boxed{1/3}$$

Answer: The rate of change is the slope of the secant line connecting the points on the curve at $x = 1$ and $x = 4$.
From the graph, we identify the coordinates: $A(1, 1)$ and $B(4, 2)$.

$$\text{Rate of change} = \frac{\Delta y}{\Delta x} = \frac{2-1}{4-1} = \frac{1}{3}$$



Ex 8: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 0$ to $x = \pi$ graphically.

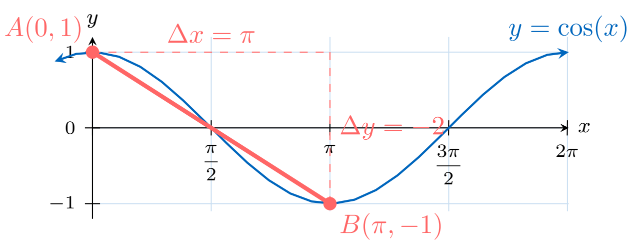
$$\text{Rate of change} = \boxed{-2/\pi}$$

Answer: The rate of change is the slope of the secant line connecting the points on the curve at $x = 0$ and $x = \pi$.
From the graph, we can identify the coordinates of these two points:

- At $x = 0$, the value is $f(0) = \cos(0) = 1$. So we have the point $A(0, 1)$.
- At $x = \pi$, the value is $f(\pi) = \cos(\pi) = -1$. So we have the point $B(\pi, -1)$.

We can now calculate the slope of the secant line (AB).

$$\begin{aligned}\text{Rate of change} = \text{Slope} &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_B - y_A}{x_B - x_A} \\ &= \frac{-1-1}{\pi - 0} \\ &= -\frac{2}{\pi}\end{aligned}$$



A.1.3 MODELING WITH RATES OF CHANGE

Ex 9: An athlete completes a 100-meter race in 20 seconds. At the start of the race ($t = 0$ s), his distance from the starting line was 0 m. At the finish line ($t = 20$ s), his distance was 100 m.
Calculate his average speed (the rate of change of distance with respect to time) over the course of the race.

$$\text{Average speed} = \boxed{5}\ \text{m/s}$$

Answer: The average speed is the rate of change of distance with respect to time. Let $d(t)$ be the distance in meters at time t in seconds. We are given the points $(t_1, d_1) = (0, 0)$ and $(t_2, d_2) = (20, 100)$.
We use the formula for the rate of change:

$$\begin{aligned}\text{Average speed} &= \frac{\text{change in distance}}{\text{change in time}} \\ &= \frac{d(20) - d(0)}{20 - 0} \\ &= \frac{100\text{ m} - 0\text{ m}}{20\text{ s} - 0\text{ s}} \\ &= \frac{100}{20}\ \text{m/s} \\ &= 5\ \text{m/s}\end{aligned}$$

The athlete's average speed during the race was 5 meters per second.

Ex 10: A company's profit is recorded over a 5-year period. At the start of the period ($t = 0$ years), the profit was \$20,000. After 5 years ($t = 5$ years), the profit was \$80,000.
Calculate the average rate of change of profit (in dollars per year) over this period.

$$\text{Average rate of change} = \boxed{12000}\ \$/\text{year}$$

Answer: The average rate of change of profit is the total change in profit divided by the total change in time. Let $P(t)$ be the profit in dollars at time t in years. We are given the points $(t_1, P_1) = (0, 20000)$ and $(t_2, P_2) = (5, 80000)$.
We use the formula for the rate of change:

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{change in profit}}{\text{change in time}} \\ &= \frac{P(5) - P(0)}{5 - 0} \\ &= \frac{80000\ \$ - 20000\ \$}{5\text{ years} - 0\text{ years}} \\ &= \frac{60000}{5}\ \$/\text{year} \\ &= 12000\ \$/\text{year}\end{aligned}$$

The company's profit grew at an average rate of \$12,000 per year.

Ex 11: At 8:00 AM ($t = 0$ hours), the temperature in a room is 15°C. By noon ($t = 4$ hours), the temperature has risen to 25°C.
Calculate the average rate of change of temperature (in degrees Celsius per hour) during this time.

$$\text{Average rate of change} = \boxed{2.5}\ {}^\circ\text{C/hour}$$

Answer: The average rate of change of temperature is the total change in temperature divided by the time elapsed. Let $T(t)$ be the temperature in °C at time t in hours. We have the points $(t_1, T_1) = (0, 15)$ and $(t_2, T_2) = (4, 25)$.
We use the formula for the rate of change:

$$\begin{aligned}
\text{Average rate of change} &= \frac{\text{change in temperature}}{\text{change in time}} \\
&= \frac{T(4) - T(0)}{4 - 0} \\
&= \frac{25°\text{C} - 15°\text{C}}{4 \text{ hours} - 0 \text{ hours}} \\
&= \frac{10}{4} \ °\text{C/hour} \\
&= 2.5 \ °\text{C/hour}
\end{aligned}$$

The temperature increased at an average rate of 2.5 degrees Celsius per hour.

Ex 12: A biologist is monitoring a cell culture. At the start of the experiment ($t = 0$ days), the population is 500 cells. After 10 days, the population has grown to 4500 cells.
Calculate the average growth rate of the cell culture (in cells per day).

$$\text{Average growth rate} = \boxed{400} \text{ cells/day}$$

Answer: The average growth rate is the rate of change of the population with respect to time. Let $P(t)$ be the number of cells at time t in days. We are given the points $(t_1, P_1) = (0, 500)$ and $(t_2, P_2) = (10, 4500)$.
We use the formula for the rate of change:

$$\begin{aligned}
\text{Average growth rate} &= \frac{\text{change in population}}{\text{change in time}} \\
&= \frac{P(10) - P(0)}{10 - 0} \\
&= \frac{4500 \text{ cells} - 500 \text{ cells}}{10 \text{ days} - 0 \text{ days}} \\
&= \frac{4000}{10} \text{ cells/day} \\
&= 400 \text{ cells/day}
\end{aligned}$$

The cell population grew at an average rate of 400 cells per day.

A.1.4 MODELING WITH RATES OF CHANGE

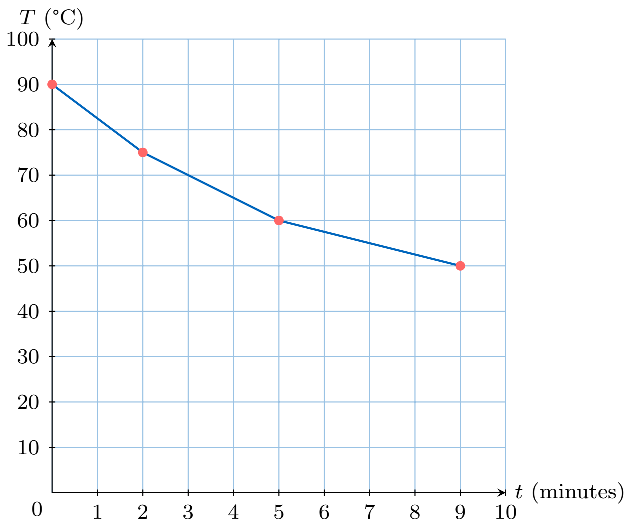
Ex 13: The temperature, T, of a cup of coffee is recorded at various times, t, after it is poured. The data is shown in the table below.

t (minutes)	0	2	5	9
T (°C)	90	75	60	50

1. Find the average rate of change of the temperature:
   (a) between $t = 0$ and $t = 2$ minutes.
   (b) between $t = 2$ and $t = 5$ minutes.
   (c) between $t = 5$ and $t = 9$ minutes.
2. What do these rates of change suggest about how the coffee is cooling?

Answer:

1. We calculate the average rate of change, $\frac{\Delta T}{\Delta t}$, for each interval.
   (a) From $t = 0$ to $t = 2$:
   $$\text{Rate of change} = \frac{75 - 90}{2 - 0} = \frac{-15}{2} = -7.5 \ °\text{C/min}$$
   (b) From $t = 2$ to $t = 5$:
   $$\text{Rate of change} = \frac{60 - 75}{5 - 2} = \frac{-15}{3} = -5 \ °\text{C/min}$$
   (c) From $t = 5$ to $t = 9$:
   $$\text{Rate of change} = \frac{50 - 60}{9 - 5} = \frac{-10}{4} = -2.5 \ °\text{C/min}$$
2. The negative sign indicates that the temperature is decreasing. The magnitude of the rate of change is getting smaller over time (from 7.5 to 5 to 2.5). This suggests that the coffee cools down fastest at the beginning and the rate of cooling slows down as the coffee gets closer to room temperature.



Ex 14: The population of a town is recorded over several years. The data is shown in the table below.

t (year)	2000	2005	2015	2020
P (population)	5,000	5,500	8,000	12,000

1. Find the average rate of change of the population (in people per year):
   (a) between 2000 and 2005.
   (b) between 2005 and 2015.
   (c) between 2015 and 2020.
2. What do these rates of change suggest about the town's growth?

Answer:

1. We calculate the average rate of change, $\frac{\Delta P}{\Delta t}$, for each interval.
   (a) From $t = 2000$ to $t = 2005$:
   $$\text{Rate of change} = \frac{5500 - 5000}{2005 - 2000} = \frac{500}{5} = 100 \text{ people/year}$$

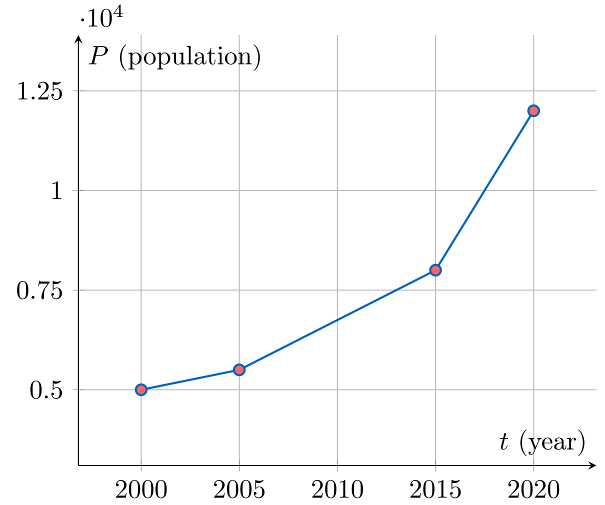
(b) From $t = 2005$ to $t = 2015$:

$$\text{Rate of change} = \frac{8000 - 5500}{2015 - 2005} = \frac{2500}{10} = 250 \text{ people/year}$$

(c) From $t = 2015$ to $t = 2020$:

$$\text{Rate of change} = \frac{12000 - 8000}{2020 - 2015} = \frac{4000}{5} = 800 \text{ people/year}$$

2. The positive rates of change indicate that the population is increasing. Since the rate of change is getting larger over time (from 100 to 250 to 800 people/year), the population growth is accelerating.



Ex 15: A swimmer completes a 400m freestyle race. Their split times are recorded every 100 meters, as shown in the table below.

t (seconds)	0	50	110	180	255
d (meters)	0	100	200	300	400

1. Find the swimmer's average speed (rate of change of distance with respect to time) for each 100m segment of the race:
   (a) from 0m to 100m.
   (b) from 100m to 200m.
   (c) from 200m to 300m.
   (d) from 300m to 400m.
2. What do these rates of change suggest about the swimmer's pacing during the race?

Answer:

1. To find speed, we calculate $\frac{\text{change in distance}}{\text{change in time}}$.

   (a) **0m to 100m:** The time taken was $50 - 0 = 50$ s.

   $$\text{Average speed} = \frac{100 \text{ m}}{50 \text{ s}} = 2 \text{ m/s}$$

   (b) **100m to 200m:** The time taken was $110 - 50 = 60$ s.

   $$\text{Average speed} = \frac{100 \text{ m}}{60 \text{ s}} \approx 1.67 \text{ m/s}$$

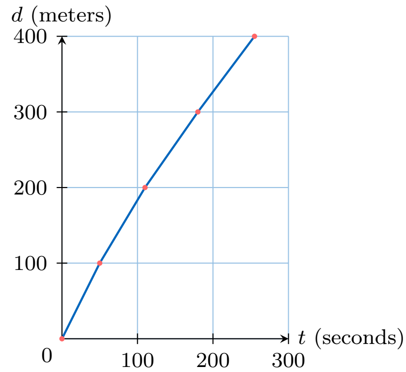
   (c) **200m to 300m:** The time taken was $180 - 110 = 70$ s.

   $$\text{Average speed} = \frac{100 \text{ m}}{70 \text{ s}} \approx 1.43 \text{ m/s}$$

   (d) **300m to 400m:** The time taken was $255 - 180 = 75$ s.

   $$\text{Average speed} = \frac{100 \text{ m}}{75 \text{ s}} \approx 1.33 \text{ m/s}$$

2. The swimmer's average speed is decreasing in each successive interval (from 2 m/s down to 1.33 m/s). This suggests the swimmer is experiencing fatigue and slowing down consistently as the race progresses.



A.2 LIMIT DEFINITION OF THE DERIVATIVE

A.2.1 CONJECTURING THE DERIVATIVE AT A POINT

Ex 16: For the function $f(x) = x^2$, find the rate of change from $x = 1$ to $x = 1 + h$:

- for $h = 1$: $\boxed{3}$
- for $h = 0.1$: $\boxed{2.1}$
- for $h = 0.01$: $\boxed{2.01}$

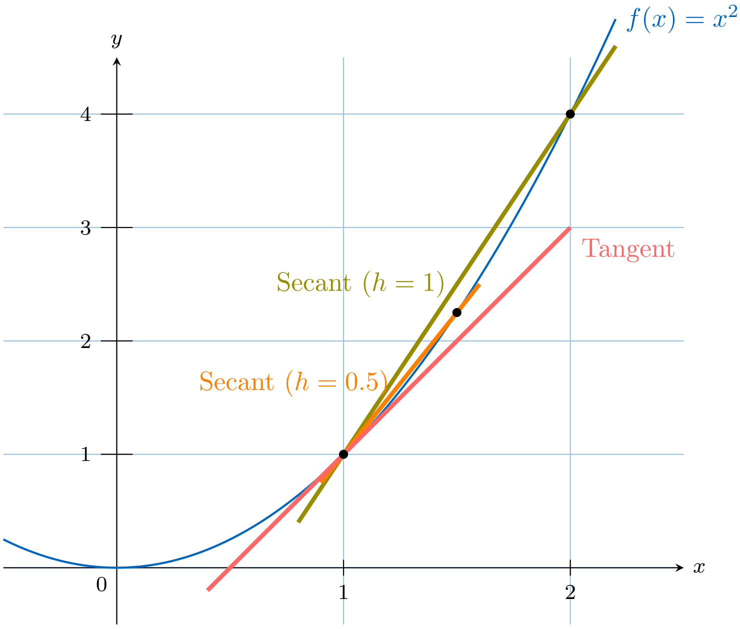
Hence, conjecture the value of the derivative at $x = 1$.

$$f'(1) = \boxed{2}$$

Answer: We use the formula for the rate of change: $\dfrac{f(1+h) - f(1)}{1 + h - 1} = \dfrac{f(1+h) - f(1)}{h}$.

- For $h = 1$: $\dfrac{f(2) - f(1)}{1} = \dfrac{2^2 - 1^2}{1} = 3$.
- For $h = 0.1$: $\dfrac{f(1.1) - f(1)}{0.1} = \dfrac{(1.1)^2 - 1^2}{0.1} = 2.1$.
- For $h = 0.01$: $\dfrac{f(1.01) - f(1)}{0.01} = \dfrac{(1.01)^2 - 1^2}{0.01} = 2.01$.

As h gets closer to 0, the rate of change gets closer to 2. We conjecture that $f'(1) = 2$.
The diagram below shows the secant lines (for $h = 1$ and $h = 0.5$) getting closer to the tangent line at $x = 1$, whose slope is 2.



Ex 17: For the function $f(x) = x^3 + 1$, find the rate of change from $x = 0$ to $x = 0 + h$:

- for $h = 1$: $\boxed{1}$
- for $h = 0.1$: $\boxed{0.01}$
- for $h = 0.01$: $\boxed{0.0001}$

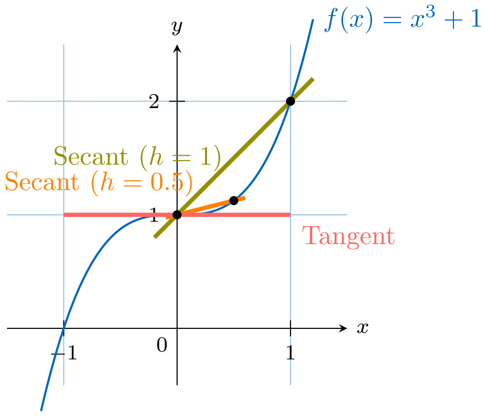
Hence, conjecture the value of the derivative at $x = 0$.

$$f'(0) = \boxed{0}$$

Answer: We use the formula for the rate of change: $\dfrac{f(0+h) - f(0)}{h} = \dfrac{f(h) - f(0)}{h}$.

- For $h = 1$: $\dfrac{f(1) - f(0)}{1} = \dfrac{(1^3 + 1) - (0^3 + 1)}{1} = 1$.
- For $h = 0.1$: $\dfrac{f(0.1) - f(0)}{0.1} = \dfrac{((0.1)^3 + 1) - 1}{0.1} = 0.01$.
- For $h = 0.01$: $\dfrac{f(0.01) - f(0)}{0.01} = \dfrac{((0.01)^3 + 1) - 1}{0.01} = 0.0001$.

As h gets closer to 0, the rate of change also gets closer to 0. We conjecture that $f'(0) = 0$.
The diagram below shows the secant lines approaching the horizontal tangent line at $x = 0$, whose slope is 0.



Ex 18: For the function $f(x) = \sqrt{x}$, find the rate of change from $x = 1$ to $x = 1 + h$. Round your answers to 4 decimal places.

- for $h = 1$: $\boxed{0.4142}$
- for $h = 0.1$: $\boxed{0.4881}$
- for $h = 0.01$: $\boxed{0.4988}$

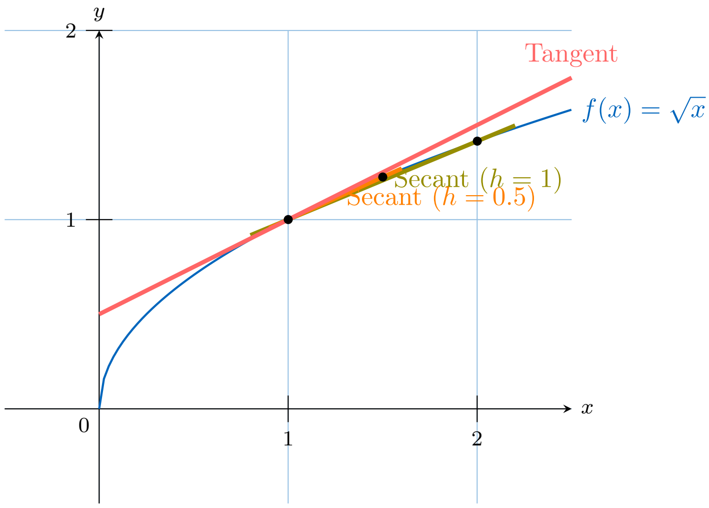
Hence, conjecture the value of the derivative at $x = 1$.

$$f'(1) = \boxed{0.5}$$

Answer: We use the formula for the rate of change: $\dfrac{f(1+h) - f(1)}{h}$.

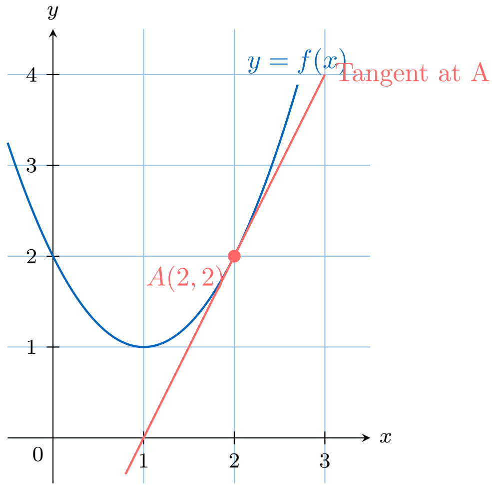
- For $h = 1$: $\dfrac{f(2) - f(1)}{1} = \dfrac{\sqrt{2} - \sqrt{1}}{1} \approx 1.4142 - 1 = 0.4142$.
- For $h = 0.1$: $\dfrac{f(1.1) - f(1)}{0.1} = \dfrac{\sqrt{1.1} - 1}{0.1} \approx \dfrac{1.04881 - 1}{0.1} = 0.4881$.
- For $h = 0.01$: $\dfrac{f(1.01) - f(1)}{0.01} = \dfrac{\sqrt{1.01} - 1}{0.01} \approx \dfrac{1.004988 - 1}{0.01} = 0.4988$.

As h gets closer to 0, the rate of change gets closer to 0.5. We conjecture that $f'(1) = 0.5$.
The diagram below shows the secant lines approaching the tangent line at $x = 1$, whose slope is 0.5.



A.2.2 FINDING THE DERIVATIVE GRAPHICALLY

Ex 19: The graph of the function $f(x) = x^2 - 2x + 2$ and its tangent line at the point $A(2, 2)$ are shown below.



Find the derivative of f at the point $x = 2$, i.e., $f'(2)$.

$$f'(2) = \boxed{2}$$

Answer: The derivative of a function at a point is equal to the slope of the tangent line at that point. We need to find the slope of the given tangent line. We can do this by identifying two points on the line and calculating the rise over run.
From the graph, the tangent line clearly passes through:

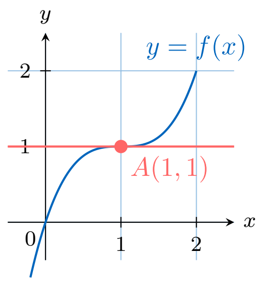
- The point of tangency, $A(2, 2)$.
- Another grid point, $B(3, 4)$.

Now, we calculate the slope (m) using these two points:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A} = \frac{4-2}{3-2} = \frac{2}{1} = 2$$

Therefore, $f'(2) = 2$.

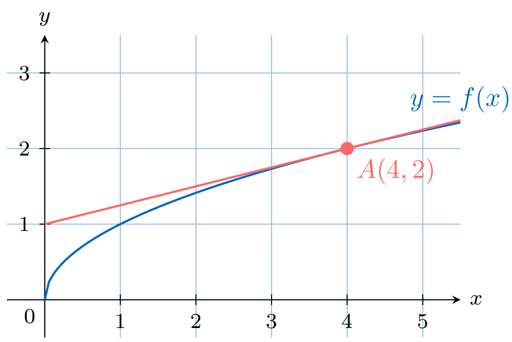
Ex 20: The graph of $f(x) = (x-1)^3 + 1$ and its tangent at $A(1, 1)$ are shown. Find $f'(1)$.



$$f'(1) = \boxed{0}$$

Answer: The derivative $f'(1)$ is the slope of the tangent line at $x = 1$.
From the graph, the tangent line at $A(1, 1)$ is a horizontal line.
The slope of any horizontal line is 0.
Therefore, $f'(1) = 0$.

Ex 21: The graph of $f(x) = \sqrt{x}$ and its tangent at $A(4, 2)$ are shown. Find $f'(4)$.



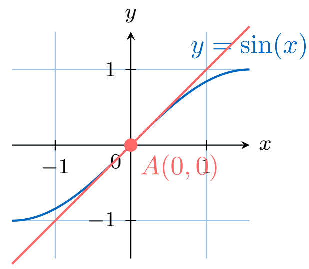
$$f'(4) = \boxed{1/4}$$

Answer: The derivative $f'(4)$ is the slope of the tangent line at $x = 4$.
From the graph, the tangent line passes through the point of tangency $A(4, 2)$ and the y-intercept at $B(0, 1)$. We calculate the slope using these two points:

$$m = \frac{\Delta y}{\Delta x} = \frac{2-1}{4-0} = \frac{1}{4}$$

Therefore, $f'(4) = \frac{1}{4}$.

Ex 22: The graph of $f(x) = \sin(x)$ and its tangent at the origin $A(0, 0)$ are shown. Find $f'(0)$.



$$f'(0) = \boxed{1}$$

Answer: The derivative $f'(0)$ is the slope of the tangent line at $x = 0$.
The tangent line is the line $y = x$, which passes through the origin $(0, 0)$ and the point $(1, 1)$.
The slope of this line is:

$$m = \frac{\Delta y}{\Delta x} = \frac{1-0}{1-0} = 1$$

Therefore, $f'(0) = 1$.

A.2.3 FINDING THE DERIVATIVE AT A POINT

Ex 23: Use the definition of the derivative (first principles) to find the derivative of $f(x) = 2x - 1$ at the point $x = 1$.

$$f'(1) = \boxed{2}$$

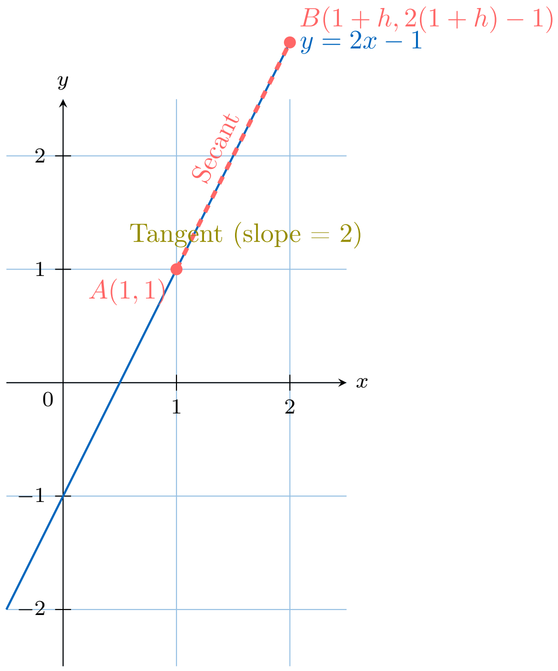
Answer: We evaluate the limit of the rate of change as $h \to 0$.

$$\begin{aligned}
\frac{f(1+h) - f(1)}{h} &= \frac{[2(1+h) - 1] - [2(1) - 1]}{h} \\
&= \frac{(2 + 2h - 1) - (1)}{h} \\
&= \frac{1 + 2h - 1}{h} \\
&= \frac{2h}{h} \\
&= 2 \quad (\text{for } h \neq 0)
\end{aligned}$$

The expression is constant, so the limit is straightforward:

$$f'(1) = \lim_{h \to 0} 2 = 2$$

The derivative at $x = 1$ is 2. This is expected, as the graph of the linear function $f(x) = 2x - 1$ is a straight line with a constant slope of 2. The tangent line at any point is the line itself.



Ex 24: Use the definition of the derivative (first principles) to find the derivative of $f(x) = x^2$ at the point $x = 1$.

$$f'(1) = \boxed{2}$$

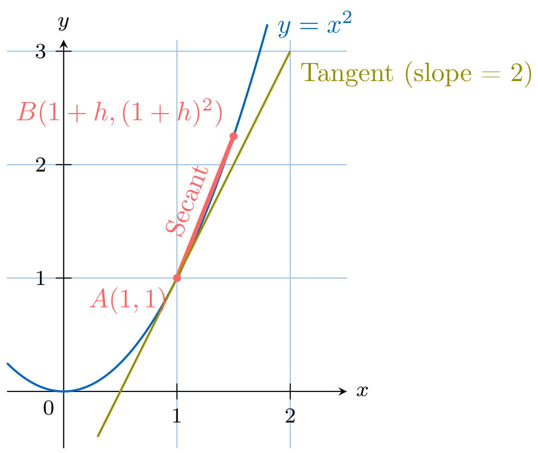
Answer: We evaluate the limit of the rate of change as $h \to 0$.

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{(1+h)^2-1^2}{h}\\ &= \frac{(1+2h+h^2)-1}{h}\\ &= \frac{2h+h^2}{h}\\ &= \frac{h(2+h)}{h}\\ &= 2+h \quad (\text{for } h \neq 0)\end{aligned}$$

Now, we take the limit:

$$f'(1) = \lim_{h\to 0}(2+h) = 2$$

The derivative at $x = 1$ is 2. This means the slope of the tangent line to the graph of $f(x) = x^2$ at the point $(1, 1)$ is 2. The diagram below shows how the slope of the secant line from $(1, 1)$ to $(1+h, f(1+h))$ approaches the slope of the tangent as h gets smaller.



Ex 25: Use the definition of the derivative (first principles) to find the derivative of $f(x) = \sqrt{x}$ at the point $x = 1$.

$$f'(1) = \boxed{1/2}$$

Answer: We evaluate the limit of the rate of change as $h \to 0$.

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{1+h}-\sqrt{1}}{h} = \frac{\sqrt{1+h}-1}{h}$$

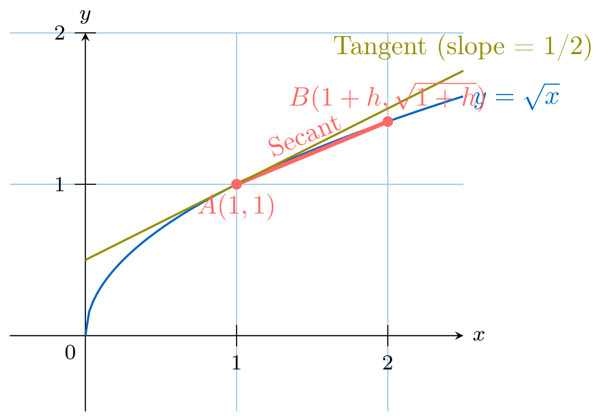
Direct substitution gives $\frac{0}{0}$. We multiply the numerator and denominator by the conjugate, $\sqrt{1+h}+1$.

$$\begin{aligned}\frac{\sqrt{1+h}-1}{h} &= \frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\\ &= \frac{\left(\sqrt{1+h}\right)^2-1^2}{h(\sqrt{1+h}+1)}\\ &= \frac{1+h-1}{h(\sqrt{1+h}+1)}\\ &= \frac{h}{h(\sqrt{1+h}+1)}\\ &= \frac{1}{\sqrt{1+h}+1} \quad (\text{for } h \neq 0)\end{aligned}$$

Now, we take the limit:

$$f'(1) = \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

The derivative at $x = 1$ is $1/2$. This means the slope of the tangent line to the graph of $f(x) = \sqrt{x}$ at the point $(1, 1)$ is $1/2$.



A.3 DERIVATIVE FUNCTION

A.3.1 FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

Ex 26: For the function $f(x) = \frac{x}{2}$, find the derivative function $f'(x)$ using first principles.

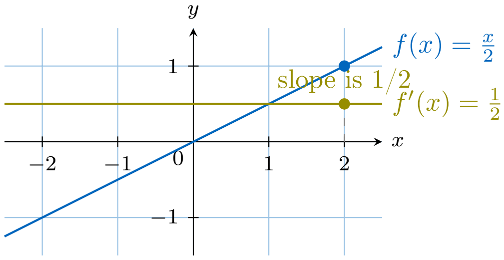
Answer: We evaluate the limit of the rate of change for a general point x.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{x+h}{2}-\frac{x}{2}}{h}\\ &= \frac{\frac{x+h-x}{2}}{h}\\ &= \frac{\frac{h}{2}}{h}\\ &= \frac{1}{2} \quad (\text{for } h \neq 0)\end{aligned}$$

The expression is constant, so the limit is:

$$f'(x) = \lim_{h\to 0}\frac{1}{2} = \frac{1}{2}$$

The derivative function is $f'(x) = \frac{1}{2}$.
The graph below shows the original function $f(x) = \frac{x}{2}$ (a line with slope 1/2) and its derivative function $f'(x) = \frac{1}{2}$ (a horizontal line). The derivative is constant because the slope of the original function is the same at every point.



Ex 27: For the function $f(x) = x^2$, find the derivative function $f'(x)$ using first principles.

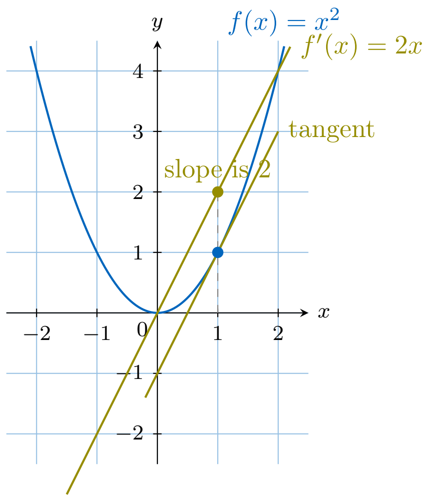
Answer: We evaluate the limit of the rate of change for a general point x.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-x^2}{h}\\ &= \frac{x^2+2xh+h^2-x^2}{h}\\ &= \frac{2xh+h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x+h \quad (\text{for } h \neq 0)\end{aligned}$$

Now, we take the limit as $h \to 0$:

$$f'(x) = \lim_{h\to 0}(2x+h) = 2x$$

The derivative function is $f'(x) = 2x$.
The graph below shows the original function $f(x) = x^2$ (a parabola) and its derivative function $f'(x) = 2x$ (a line). For any x-value, the y-value of the line gives the slope of the tangent to the parabola at that x-value. For example, at $x = 1$, the slope of the parabola's tangent is $f'(1) = 2(1) = 2$.



Ex 28: For the function $f(x) = \dfrac{1}{x}$, find the derivative function $f'(x)$ using first principles.

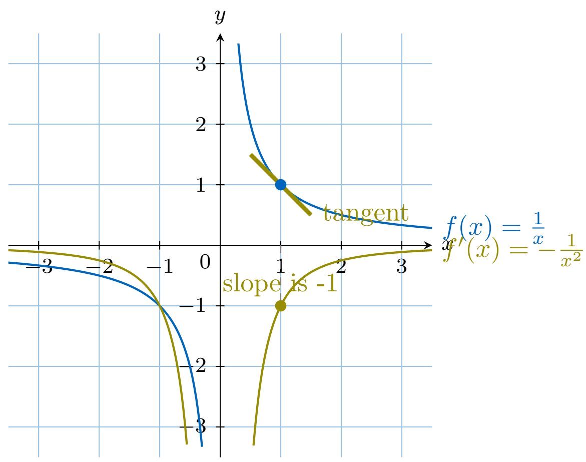
Answer: We evaluate the limit of the rate of change for a general point $x \neq 0$.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \frac{\frac{x-(x+h)}{x(x+h)}}{h}\\ &= \frac{-h}{h\cdot x(x+h)}\\ &= \frac{-1}{x(x+h)} \quad (\text{for } h \neq 0)\end{aligned}$$

Now, we take the limit as $h \to 0$:

$$f'(x) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

The derivative function is $f'(x) = -\dfrac{1}{x^2}$.
The graph below shows the original function $f(x) = \frac{1}{x}$ and its derivative $f'(x) = -\frac{1}{x^2}$. Note that the derivative is always negative, which corresponds to the fact that the original function is always decreasing.



Ex 29: For the function $f(x) = 3$, find the derivative function $f'(x)$ using first principles.

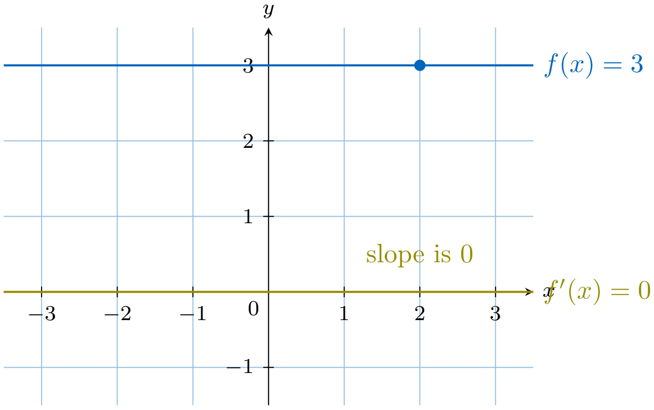
Answer: We evaluate the limit of the rate of change for a general point x.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3-3}{h}\\ &= \frac{0}{h}\\ &= 0 \quad (\text{for } h \neq 0)\end{aligned}$$

The expression is constant, so we take the limit as $h \to 0$:

$$f'(x) = \lim_{h\to 0} 0 = 0$$

The derivative function is $f'(x) = 0$.
The graph below shows the original function $f(x) = 3$ (a horizontal line) and its derivative function $f'(x) = 0$ (the x-axis). The derivative is 0 because the slope of the horizontal line is 0 at every point.



Ex 30: For the function $f(x) = \sqrt{x}$, find the derivative function $f'(x)$ using first principles.

Answer: We evaluate the limit of the rate of change for a general point $x > 0$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Direct substitution gives $\frac{0}{0}$. We must simplify by multiplying the numerator and denominator by the conjugate, $\sqrt{x+h} + \sqrt{x}$.

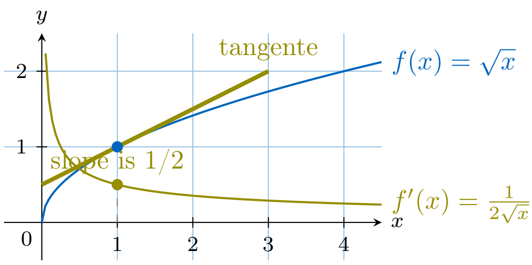
$$\begin{aligned}\frac{\sqrt{x+h} - \sqrt{x}}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \quad (\text{for } h \neq 0)\end{aligned}$$

Now, we take the limit as $h \to 0$:

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

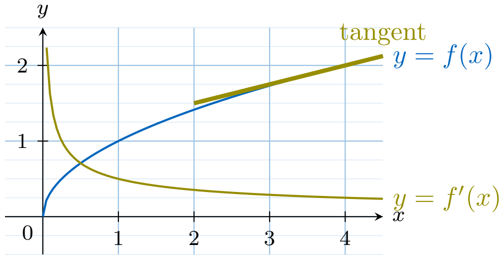
The derivative function is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

The graph below shows $f(x) = \sqrt{x}$ and its derivative $f'(x) = \frac{1}{2\sqrt{x}}$. The derivative is always positive, corresponding to the fact that $\sqrt{x}$ is always increasing. Also, as x increases, the slope of $\sqrt{x}$ decreases, which is reflected in the decreasing values of the derivative function.



A.3.2 INTERPRETING THE GRAPH OF THE DERIVATIVE

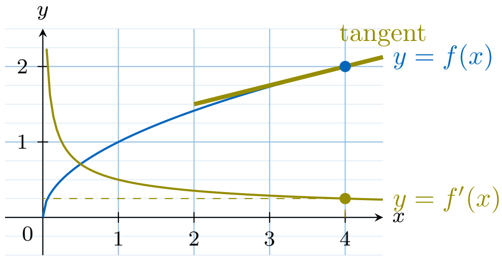
Ex 31: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = 4$.

Slope at $x = 4$ is $\boxed{1/4}$

Answer: The value of the derivative function $f'(x)$ at a specific point gives the slope of the tangent to the original function $f(x)$ at that same point.
To find the slope of the tangent at $x = 4$, we need to find the value of $f'(4)$ by reading the y-coordinate on the graph of $y = f'(x)$ when $x = 4$.

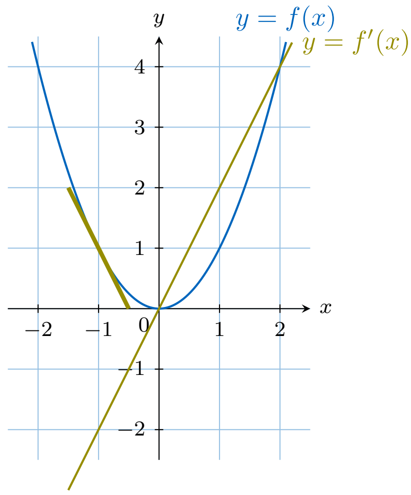


From the graph, we can see that the point $(4, 0.25)$ lies on the curve of $f'(x)$.

Therefore, $f'(4) = 0.25 = \dfrac{1}{4}$.

The slope of the tangent to $f(x)$ at $x = 4$ is $\dfrac{1}{4}$.

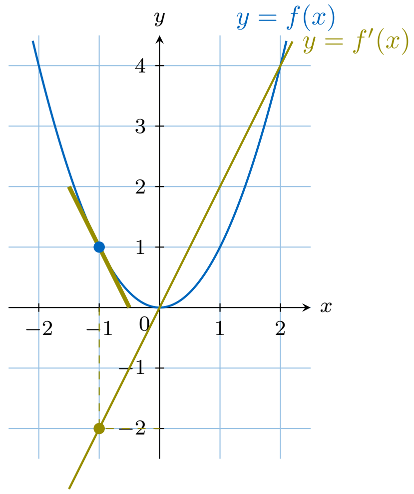
Ex 32: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = -1$.

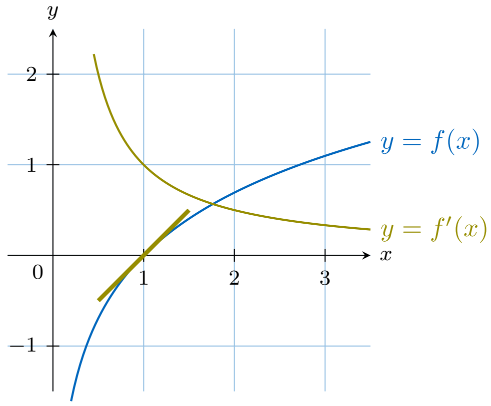
Slope at $x = -1$ is $\boxed{-2}$

Answer: The value of the derivative function $f'(x)$ at a specific point gives the slope of the tangent to the original function $f(x)$ at that same point.
To find the slope of the tangent at $x = -1$, we need to find the value of $f'(-1)$ by reading the y-coordinate on the graph of $y = f'(x)$ when $x = -1$.



From the graph, we can see that the point $(-1, -2)$ lies on the curve of $f'(x)$.
Therefore, $f'(-1) = -2$.
The slope of the tangent to $f(x)$ at $x = -1$ is -2.

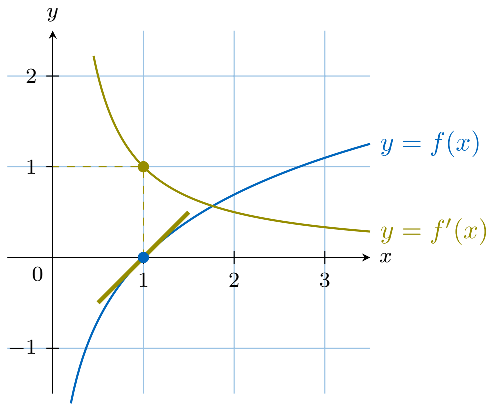
Ex 33: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = 1$.

Slope at $x = 1$ is $\boxed{1}$

Answer: The value of the derivative function $f'(x)$ at a specific point gives the slope of the tangent to the original function $f(x)$ at that same point.
To find the slope of the tangent at $x = 1$, we need to find the value of $f'(1)$ by reading the y-coordinate on the graph of $y = f'(x)$ when $x = 1$.



From the graph, we can see that the point $(1, 1)$ lies on the curve of $f'(x)$.
Therefore, $f'(1) = 1$. The slope of the tangent to $f(x)$ at $x = 1$ is 1.

A.3.3 FINDING THE TANGENT SLOPE USING THE DERIVATIVE FUNCTION

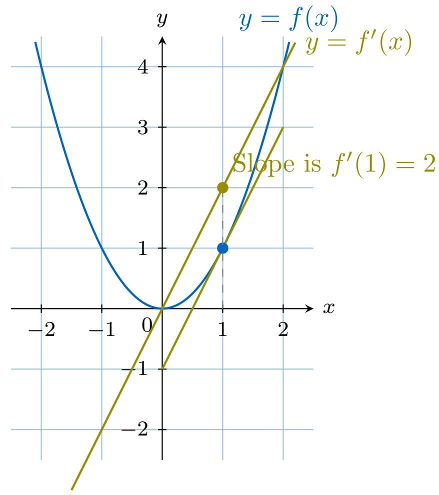
Ex 34: The derivative of a function $f(x)$ is given by $f'(x) = 2x$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = 1$.

Slope at $x = 1 = \boxed{2}$

Answer: The slope of the tangent to the graph of $f(x)$ at a point $x = a$ is given by the value of the derivative, $f'(a)$.
We are given the derivative function $f'(x) = 2x$. To find the slope of the tangent at $x = 1$, we simply need to evaluate $f'(1)$.

$$\begin{aligned} f'(1) &= 2(1) \\ &= 2 \end{aligned}$$

Therefore, the slope of the tangent to the graph of $f(x)$ at $x = 1$ is 2.
The graph below shows the derivative function $f'(x) = 2x$. The y-value of this function at $x = 1$ is $f'(1) = 2$. This value represents the slope of the tangent to the graph of any function of the form $f(x) = x^2 + C$ at $x = 1$.



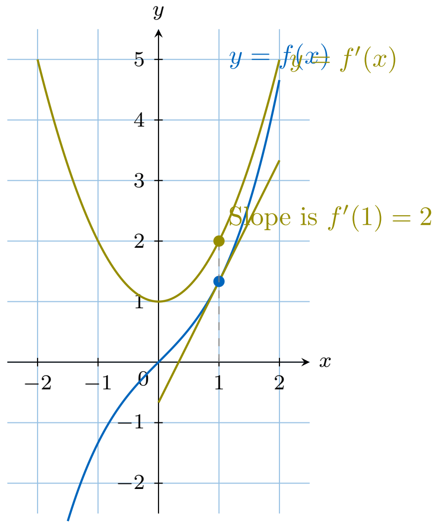
Ex 35: The derivative of a function $f(x)$ is given by $f'(x) = x^2 + 1$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = 1$.

Slope at $x = 1 = \boxed{2}$

Answer: The slope of the tangent to the graph of $f(x)$ at a point $x = a$ is given by the value of the derivative, $f'(a)$.
We are given the derivative function $f'(x) = x^2 + 1$. To find the slope of the tangent at $x = 1$, we simply need to evaluate $f'(1)$.

$$\begin{aligned} f'(1) &= (1)^2 + 1 \\ &= 2 \end{aligned}$$

Therefore, the slope of the tangent to the graph of $f(x)$ at $x = 1$ is 2.
The graph below shows the derivative function $f'(x) = x^2 + 1$. The y-value of this function at $x = 1$ is $f'(1) = 2$. This value represents the slope of the tangent to the graph of any function of the form $f(x) = \frac{1}{3}x^3 + x + C$ at $x = 1$.



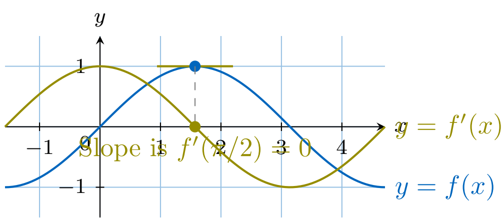
Ex 36: The derivative of a function $f(x)$ is given by $f'(x) = \cos(x)$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = \dfrac{\pi}{2}$.

$$\text{Slope at } x = \frac{\pi}{2} = \boxed{0}$$

Answer: The slope of the tangent to the graph of $f(x)$ at a point $x = a$ is given by the value of the derivative, $f'(a)$.
We are given the derivative function $f'(x) = \cos(x)$. To find the slope of the tangent at $x = \dfrac{\pi}{2}$, we simply need to evaluate $f'(\pi/2)$.

$$\begin{aligned} f'(\pi/2) &= \cos(\pi/2) \\ &= 0 \end{aligned}$$

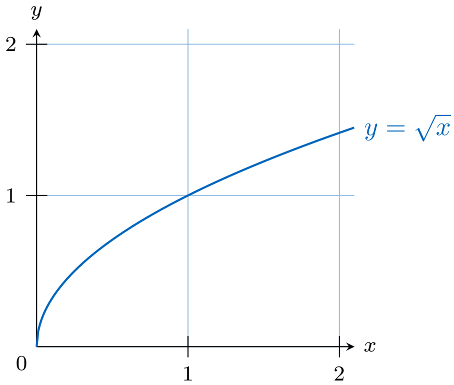
Therefore, the slope of the tangent to the graph of $f(x)$ at $x = \pi/2$ is 0.
The graph below shows the derivative function $f'(x) = \cos(x)$. The y-value of this function at $x = \pi/2$ is $f'(\pi/2) = 0$. This value represents the slope of the tangent to the graph of any function of the form $f(x) = \sin(x) + C$ at $x = \pi/2$.



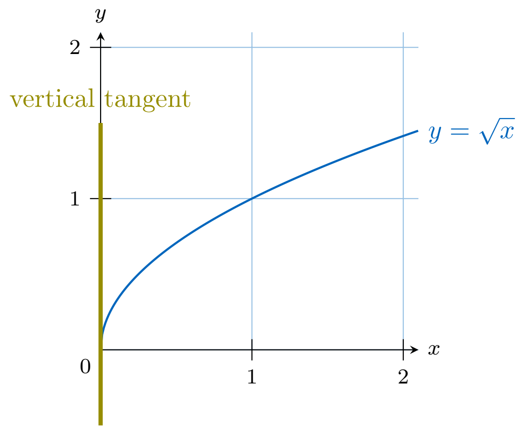
A.4 CONDITIONS OF DIFFERENTIABILITY

A.4.1 IDENTIFYING DIFFERENTIABILITY FROM A GRAPH

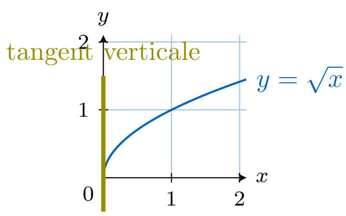
Ex 37: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.



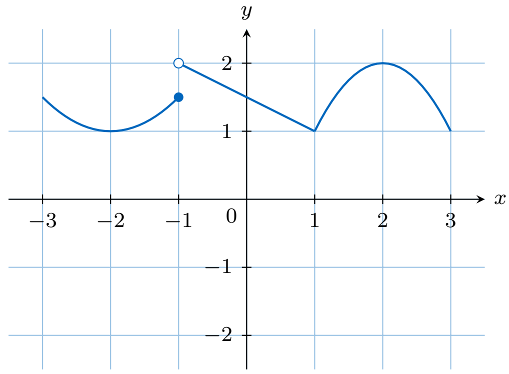
Answer:



The function is not differentiable at $\boldsymbol{x = 0}$. There is a **vertical tangent**.

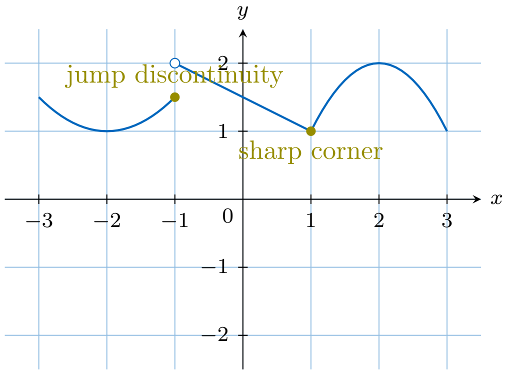


Ex 38: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.

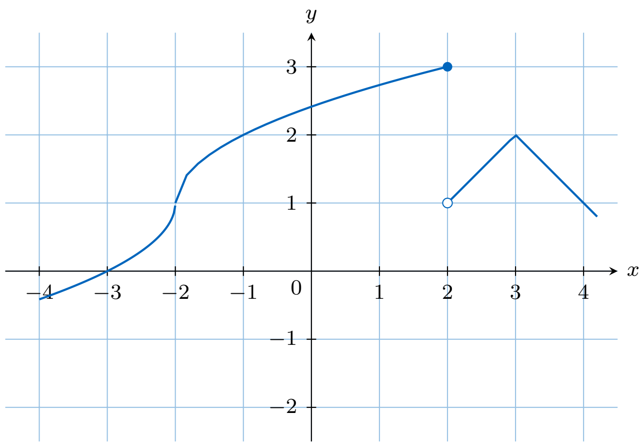


Answer: The function is not differentiable at two points:

- At $\boldsymbol{x = -1}$, there is a **jump discontinuity**. Since the function is not continuous here, it cannot be differentiable.
- At $\boldsymbol{x = 1}$, there is a **sharp corner**. The slope of the line segment from the left is -0.5, while the tangent to the parabola from the right has a slope of $-2(1-2) = 2$. Since the left and right slopes are not equal, the function is not differentiable at this point.

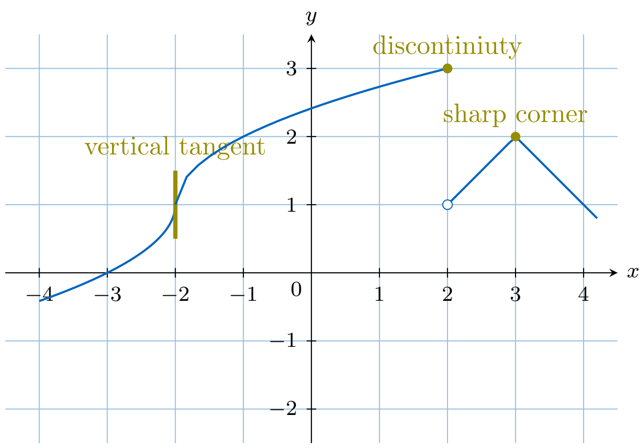


Ex 39: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.



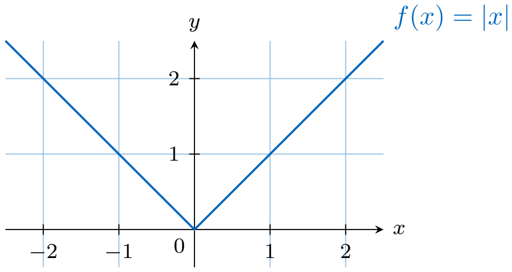
Answer: The function is not differentiable at three points:

- At $\boldsymbol{x = -2}$, the graph has a **vertical tangent**. The slope of the tangent line is infinite at this point.
- At $\boldsymbol{x = 2}$, there is a **jump discontinuity**. The function is not continuous, so it cannot be differentiable.
- At $\boldsymbol{x = 3}$, there is a **sharp corner**. The slope from the left is -1, while the slope from the right is 1.



A.4.2 PROVING NON-DIFFERENTIABILITY AT A POINT

Ex 40: Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

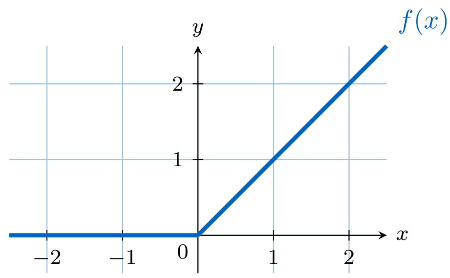


Answer: To be differentiable at $x = 0$, the limit *displaystyle* $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ must exist. We check the limits from the left and right.

- **Limit from the right ($h \to 0^+$):**
  For $h > 0$, $|h| = h$.
  $$\frac{|h| - 0}{h} = \frac{h}{h} = 1 \xrightarrow[h\to 0]{} 1$$
- **Limit from the left ($h \to 0^-$):**
  For $h < 0$, $|h| = -h$.
  $$\frac{|h| - 0}{h} = \frac{-h}{h} = -1 \xrightarrow[h\to 0]{} -1$$

Since the limit from the left (-1) does not equal the limit from the right (1), the derivative does not exist at $x = 0$.

Ex 41: Show that the function $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$ is not differentiable at $x = 0$.



Answer: To be differentiable at $x = 0$, the limit $\displaystyle\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ must exist. We check the limits from the left and right.

- **Limit from the right ($h \to 0^+$):**
  For $h > 0$, $f(h) = h$.
  $$\frac{f(h) - f(0)}{h} = \frac{h - 0}{h} = 1 \xrightarrow[h\to 0]{} 1$$
- **Limit from the left ($h \to 0^-$):**
  For $h < 0$, $f(h) = 0$.
  $$\frac{f(h) - f(0)}{h} = \frac{0 - 0}{h} = 0 \xrightarrow[h\to 0]{} 0$$

Since the limit from the left (0) does not equal the limit from the right (1), the derivative does not exist at $x = 0$.

B RULES OF DIFFERENTIATION

B.1 BASIC RULES AND POWER FUNCTIONS

B.1.1 PROVING BASIC RULES AND POWER FUNCTIONS

Ex 42: Prove that: if $f(x) = k$, then $f'(x) = 0$, where k is a constant.

Answer: We evaluate the limit of the rate of change.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{k-k}{h}\\ &= \frac{0}{h}\\ &= 0 \quad (\text{for } h \neq 0)\\ &\xrightarrow[h\to 0]{} 0\end{aligned}$$

So $f'(x) = 0$.

Ex 43: Prove that: if $f(x) = x^2$, then $f'(x) = 2x$.

Answer: We evaluate the limit of the rate of change.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-x^2}{h}\\ &= \frac{x^2+2xh+h^2-x^2}{h}\\ &= \frac{2xh+h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x+h \quad (\text{for } h \neq 0)\\ &\xrightarrow[h\to 0]{} 2x\end{aligned}$$

So $f'(x) = 2x$.

Ex 44: Prove that: if $f(x) = ku(x)$, then $f'(x) = ku'(x)$, where k is a constant.

Answer: We evaluate the limit of the rate of change.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{ku(x+h)-ku(x)}{h}\\ &= \frac{k[u(x+h)-u(x)]}{h}\\ &= k\cdot\frac{u(x+h)-u(x)}{h}\\ &\xrightarrow[h\to 0]{} k\cdot u'(x) \quad (\text{constant multiple of limits}).\end{aligned}$$

So $f'(x) = ku'(x)$.

Ex 45: Prove that: if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

Answer: We evaluate the limit of the rate of change.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}\\ &= \frac{[u(x+h)-u(x)]+[v(x+h)-v(x)]}{h}\\ &= \frac{u(x+h)-u(x)}{h}+\frac{v(x+h)-v(x)}{h}\\ &\xrightarrow[h\to 0]{} u'(x)+v'(x) \quad (\text{sum of limits}).\end{aligned}$$

So $f'(x) = u'(x) + v'(x)$.

B.1.2 APPLYING THE POWER RULE

Ex 46: Find the derivative of $f(x) = x^4$.

$$f'(x) = \boxed{4x^3}$$

Answer: The function is $f(x) = x^4$. This is a power function of the form x^n with $n = 4$.We apply the Power Rule, which states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^4)\\ &= 4x^{4-1}\\ &= 4x^3\end{aligned}$$

Ex 47: Find the derivative of $f(x) = x$.

$$f'(x) = \boxed{1}$$

Answer: The function is $f(x) = x$, which can be written as $f(x) = x^1$. This is a power function of the form x^n with $n = 1$.We apply the Power Rule, which states that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^1)\\ &= 1x^{1-1}\\ &= 1x^0\\ &= 1\end{aligned}$$

Ex 48: Find the derivative of $f(x) = \frac{1}{x^2}$.

$$f'(x) = \boxed{-\frac{2}{x^3}}$$

Answer: First, we write the function in the form x^n.

$$f(x) = \frac{1}{x^2} = x^{-2}$$

This is a power function with $n = -2$. We apply the Power Rule, $f'(x) = nx^{n-1}$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^{-2})\\ &= -2x^{-2-1}\\ &= -2x^{-3}\\ &= -\frac{2}{x^3}\end{aligned}$$

Ex 49: Find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \boxed{\frac{1}{2*sqrt(x)}}$$

Answer: First, we write the function in the form x^n.

$$f(x) = \sqrt{x} = x^{1/2}$$

This is a power function with $n = 1/2$. We apply the Power Rule, $f'(x) = nx^{n-1}$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^{1/2})\\ &= \frac{1}{2}x^{\frac{1}{2}-1}\\ &= \frac{1}{2}x^{-1/2}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

Ex 50: Find the derivative of $f(x) = \dfrac{1}{x}$.

$$f'(x) = \boxed{-\frac{1}{x^2}}$$

Answer: First, we write the function in the form x^n.

$$f(x) = \frac{1}{x} = x^{-1}$$

This is a power function with $n = -1$. We apply the Power Rule, $f'(x) = nx^{n-1}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{-1}) \\ &= -1x^{-1-1} \\ &= -1x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

B.1.3 DIFFERENTIATING POLYNOMIAL FUNCTIONS

Ex 51: Find the derivative of $f(x) = 3x - 2$.

$$f'(x) = \boxed{3}$$

Answer: We apply the Sum and Constant Multiple rules to differentiate the function term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x-2) \\ &= 3\frac{d}{dx}(x) - \frac{d}{dx}(2) \quad \text{(linearity)} \\ &= 3(1) - 0 \\ &= 3 \end{aligned}$$

Ex 52: Find the derivative of $f(x) = x^2 + 4x - 5$.

$$f'(x) = \boxed{2x + 4}$$

Answer: We apply the Sum and Constant Multiple rules to differentiate the function term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2 + 4x - 5) \\ &= \frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) - \frac{d}{dx}(5) \quad \text{(linearity)} \\ &= (2x) + 4(1) - 0 \\ &= 2x + 4 \end{aligned}$$

Ex 53: Find the derivative of $f(x) = 5x^3 - 2x^2 + 1$.

$$f'(x) = \boxed{15x^2 - 4x}$$

Answer: We differentiate the function term by term using linearity.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3 - 2x^2 + 1) \\ &= 5\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + \frac{d}{dx}(1) \quad \text{(linearity)} \\ &= 5(3x^2) - 2(2x) + 0 \\ &= 15x^2 - 4x \end{aligned}$$

Ex 54: Find the derivative of $f(x) = x^5 - \frac{1}{2}x^4 + 3x$.

$$f'(x) = \boxed{5x^4 - 2x^3 + 3}$$

Answer: We differentiate the function term by term using linearity.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^5 - \frac{1}{2}x^4 + 3x\right) \\ &= \frac{d}{dx}(x^5) - \frac{1}{2}\frac{d}{dx}(x^4) + 3\frac{d}{dx}(x) \quad \text{(linearity)} \\ &= (5x^4) - \frac{1}{2}(4x^3) + 3(1) \\ &= 5x^4 - 2x^3 + 3 \end{aligned}$$

B.1.4 DIFFERENTIATING FUNCTIONS WITH FRACTIONAL AND NEGATIVE EXPONENTS

Ex 55: Find the derivative of $f(x) = 2x + 3\sqrt{x}$.

$$f'(x) = \boxed{2 + \frac{3}{2\sqrt{x}}}$$

Answer: First, we rewrite the function using a power: $f(x) = 2x + 3x^{1/2}$.We now apply the Sum and Power rules term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x) + \frac{d}{dx}(3x^{1/2}) \\ &= 2 + 3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) \\ &= 2 + \frac{3}{2}x^{-1/2} \\ &= 2 + \frac{3}{2\sqrt{x}} \end{aligned}$$

Ex 56: Find the derivative of $f(x) = 3x + 2 + \frac{5}{x}$.

$$f'(x) = \boxed{3 - \frac{5}{x^2}}$$

Answer: First, we rewrite the function using a power: $f(x) = 3x + 2 + 5x^{-1}$.We now differentiate term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x) + \frac{d}{dx}(2) + \frac{d}{dx}(5x^{-1}) \\ &= 3 + 0 + 5(-1x^{-1-1}) \\ &= 3 - 5x^{-2} \\ &= 3 - \frac{5}{x^2} \end{aligned}$$

Ex 57: Find the derivative of $f(x) = 3x\sqrt{x} - 2x$.

$$f'(x) = \boxed{\frac{9\sqrt{x}}{2} - 2}$$

Answer: First, we simplify the function by combining the powers of x:

$$f(x) = 3x^1 \cdot x^{1/2} - 2x = 3x^{3/2} - 2x$$

Now we can differentiate using the Power Rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^{3/2}) - \frac{d}{dx}(2x) \\ &= 3\left(\frac{3}{2}x^{\frac{3}{2}-1}\right) - 2 \\ &= \frac{9}{2}x^{1/2} - 2 \\ &= \frac{9\sqrt{x}}{2} - 2 \end{aligned}$$

Ex 58: Find the derivative of $f(x) = 2x^2 - \dfrac{1}{5x}$.

$$f'(x) = \boxed{4x + \frac{1}{5x^2}}$$

Answer: First, we rewrite the function using a negative exponent:

$$f(x) = 2x^2 - \frac{1}{5}x^{-1}$$

Now we differentiate term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^2) - \frac{d}{dx}\left(\frac{1}{5}x^{-1}\right) \\ &= 2(2x) - \frac{1}{5}(-1x^{-1-1}) \\ &= 4x + \frac{1}{5}x^{-2} \\ &= 4x + \frac{1}{5x^2} \end{aligned}$$

B.1.5 EXPANDING BEFORE DIFFERENTIATING

Ex 59: By first simplifying the expression into a sum of terms, find the derivative of $f(x) = \dfrac{x-1}{x}$.

$$f'(x) = \boxed{1/x^2}$$

Answer: First, we simplify the function by splitting the fraction:

$$f(x) = \frac{x}{x} - \frac{1}{x} = 1 - x^{-1}$$

Now, we can differentiate this simplified expression term by term using the Constant Rule and the Power Rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1) - \frac{d}{dx}(x^{-1}) \\ &= 0 - (-1x^{-1-1}) \\ &= 0 - (-1x^{-2}) \\ &= x^{-2} = \frac{1}{x^2} \end{aligned}$$

Ex 60: By first expanding the expression, find the derivative of $f(x) = (x+1)^2$.

$$f'(x) = \boxed{2x + 2}$$

Answer: First, we expand the binomial expression:

$$f(x) = (x+1)(x+1) = x^2 + 2x + 1$$

Now, we can differentiate this polynomial term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(1) \\ &= 2x + 2 + 0 \\ &= 2x + 2 \end{aligned}$$

Ex 61: By first expanding the expression, find the derivative of $f(x) = (2x^2 - 3)^2$.

$$f'(x) = \boxed{16x^3 - 24x}$$

Answer: First, we expand the binomial expression:

$$f(x) = (2x^2 - 3)(2x^2 - 3) = 4x^4 - 6x^2 - 6x^2 + 9 = 4x^4 - 12x^2 + 9$$

Now, we can differentiate this polynomial term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(4x^4) - \frac{d}{dx}(12x^2) + \frac{d}{dx}(9) \\ &= 4(4x^3) - 12(2x) + 0 \\ &= 16x^3 - 24x \end{aligned}$$

Ex 62: By first simplifying the expression, find the derivative of $f(x) = \dfrac{2x^3 - x}{x}$.

$$f'(x) = \boxed{4x}$$

Answer: First, we simplify the function by splitting the fraction (or factoring x from the numerator). For $x \neq 0$:

$$f(x) = \frac{2x^3}{x} - \frac{x}{x} = 2x^2 - 1$$

Now, we can differentiate this simplified expression term by term.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^2) - \frac{d}{dx}(1) \\ &= 2(2x) - 0 \\ &= 4x \end{aligned}$$

B.2 CHAIN RULE

B.2.1 FORMING COMPOSITE FUNCTIONS

Ex 63: If $v(x) = x^3$ and $u(x) = 2x - 1$, find the composite function $f(x) = v(u(x))$.

$$f(x) = \boxed{(2x-1)^3}$$

Answer: We find the composite function by substituting the expression for $u(x)$ into the variable of $v(x)$.

$$\begin{aligned} f(x) &= v(u(x)) \\ &= v(2x-1) \\ &= (2x-1)^3 \end{aligned}$$

Ex 64: If $v(x) = \dfrac{2}{x}$ and $u(x) = x^2 - 1$, find the composite function $f(x) = v(u(x))$.

$$f(x) = \boxed{\frac{2}{x^2 - 1}}$$

Answer: We find the composite function by substituting the expression for $u(x)$ into the variable of $v(x)$.

$$\begin{aligned} f(x) &= v(u(x)) \\ &= v(x^2 - 1) \\ &= \frac{2}{x^2 - 1} \end{aligned}$$

Ex 65: If $v(x) = 3\sqrt{x}$ and $u(x) = x^4 + 1$, find the composite function $f(x) = v(u(x))$.

$$f(x) = \boxed{3\sqrt{x^4+1}}$$

Answer: We find the composite function by substituting the expression for $u(x)$ into the variable of $v(x)$.

$$\begin{aligned} f(x) &= v(u(x)) \\ &= v(x^4+1) \\ &= 3\sqrt{x^4+1} \end{aligned}$$

Ex 66: If $v(x) = 5x^2$ and $u(x) = 3x+2$, find the composite function $f(x) = v(u(x))$.

$$f(x) = \boxed{5(3x+2)^2}$$

Answer: We find the composite function by substituting the expression for $u(x)$ into the variable of $v(x)$.

$$\begin{aligned} f(x) &= v(u(x)) \\ &= v(3x+2) \\ &= 5(3x+2)^2 \end{aligned}$$

B.2.2 DECOMPOSING COMPOSITE FUNCTIONS

Ex 67: Decompose the function $f(x) = (2x-1)^3$ into an outer function v and an inner function u such that $f(x) = v(u(x))$.

$$\begin{aligned} u(x) &= \boxed{2x-1} \\ v(x) &= \boxed{x^3} \end{aligned}$$

Answer: The "inner" part of the function is the expression inside the parentheses, and the "outer" part is the operation being performed on it.

- The inner function is $u(x) = 2x-1$.
- The outer function takes an input and cubes it, so $v(x) = x^3$.

Check: $v(u(x)) = v(2x-1) = (2x-1)^3 = f(x)$.

Ex 68: Decompose the function $f(x) = \dfrac{2}{x^2-1}$ into an outer function v and an inner function u.

$$\begin{aligned} u(x) &= \boxed{x^2-1} \\ v(x) &= \boxed{2/x} \end{aligned}$$

Answer: The expression in the denominator is the input to the reciprocal function.

- The inner function is $u(x) = x^2-1$.
- The outer function takes an input and places it in the denominator under 2, so $v(x) = \dfrac{2}{x}$.

Check: $v(u(x)) = v(x^2-1) = \dfrac{2}{x^2-1} = f(x)$.

Ex 69: Decompose the function $f(x) = 3\sqrt{x^4+1}$ into an outer function v and an inner function u.

$$\begin{aligned} u(x) &= \boxed{x^4+1} \\ v(x) &= \boxed{3*sqrt(x)} \end{aligned}$$

Answer: The expression under the square root is the inner function.

- The inner function is $u(x) = x^4+1$.
- The outer function takes the square root of an input and multiplies by 3, so $v(x) = 3\sqrt{x}$.

Check: $v(u(x)) = v(x^4+1) = 3\sqrt{x^4+1} = f(x)$.

Ex 70: Decompose the function $f(x) = e^{2x-5}$ into an outer function v and an inner function u.

$$\begin{aligned} u(x) &= \boxed{2x-5} \\ v(x) &= \boxed{e^x} \end{aligned}$$

Answer: The expression in the exponent is the inner function.

- The inner function is $u(x) = 2x-5$.
- The outer function is the natural exponential function, so $v(x) = e^x$.

Check: $v(u(x)) = v(2x-5) = e^{2x-5} = f(x)$.

B.2.3 DIFFERENTIATING WITH THE CHAIN RULE

Ex 71: Find the derivative of $f(x) = \frac{1}{x^2+1}$.

$$f'(x) = \boxed{-\frac{2x}{(x^2+1)^2}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = x^{-1}$ and the inner function be $u(x) = x^2+1$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = -x^{-2}$ and $u'(x) = 2x$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= -(u(x))^{-2} \cdot (2x) \\ &= -(x^2+1)^{-2} \cdot (2x) \\ &= -\frac{2x}{(x^2+1)^2} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = u^{-1}$ and $u = x^2+1$, the derivatives are $\frac{dy}{du} = -u^{-2}$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (-u^{-2}) \cdot (2x) \\ &= -(x^2+1)^{-2} \cdot (2x) \\ &= -\frac{2x}{(x^2+1)^2} \end{aligned}$$

Ex 72: Find the derivative of $f(x) = 2\sqrt{2x-1}$.

$$f'(x) = \boxed{\frac{2}{\sqrt{2x-1}}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = 2\sqrt{x}$ and the inner function be $u(x) = 2x - 1$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = \frac{1}{\sqrt{x}}$ and $u'(x) = 2$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{\sqrt{u(x)}} \cdot (2) \\ &= \frac{1}{\sqrt{2x-1}} \cdot (2) \\ &= \frac{2}{\sqrt{2x-1}} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = 2\sqrt{u}$ and $u = 2x - 1$, the derivatives are $\frac{dy}{du} = \frac{1}{\sqrt{u}}$ and $\frac{du}{dx} = 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{\sqrt{u}} \cdot (2) \\ &= \frac{1}{\sqrt{2x-1}} \cdot (2) \\ &= \frac{2}{\sqrt{2x-1}} \end{aligned}$$

Ex 73: Find the derivative of $f(x) = \sqrt[3]{x^3 + 8}$.

$$f'(x) = \boxed{\frac{x^2}{(x^3+8)^{2/3}}}$$

Answer:

- **Using prime notation:**
  First, rewrite the function as $f(x) = (x^3 + 8)^{1/3}$.
  Let the outer function be $v(x) = x^{1/3}$ and the inner function be $u(x) = x^3 + 8$. The derivatives are $v'(x) = \frac{1}{3}x^{-2/3}$ and $u'(x) = 3x^2$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{3}(u(x))^{-2/3} \cdot (3x^2) \\ &= \frac{1}{3}(x^3+8)^{-2/3} \cdot (3x^2) \\ &= \frac{3x^2}{3(x^3+8)^{2/3}} = \frac{x^2}{(x^3+8)^{2/3}} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = u^{1/3}$ and $u = x^3+8$, the derivatives are $\frac{dy}{du} = \frac{1}{3}u^{-2/3}$ and $\frac{du}{dx} = 3x^2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{1}{3}u^{-2/3}\right) \cdot (3x^2) \\ &= \frac{1}{3}(x^3+8)^{-2/3} \cdot (3x^2) \\ &= \frac{x^2}{(x^3+8)^{2/3}} \end{aligned}$$

Ex 74: Find the derivative of $f(x) = \dfrac{4}{\sqrt{x^2+9}}$.

$$f'(x) = \boxed{-\frac{4x}{(x^2+9)^{3/2}}}$$

Answer:

- **Using prime notation:**
  First, rewrite the function as $f(x) = 4(x^2 + 9)^{-1/2}$.
  Let the outer function be $v(x) = 4x^{-1/2}$ and the inner function be $u(x) = x^2 + 9$. The derivatives are $v'(x) = 4(-\frac{1}{2}x^{-3/2}) = -2x^{-3/2}$ and $u'(x) = 2x$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= -2(u(x))^{-3/2} \cdot (2x) \\ &= -2(x^2+9)^{-3/2} \cdot (2x) \\ &= -4x(x^2+9)^{-3/2} = -\frac{4x}{(x^2+9)^{3/2}} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = 4u^{-1/2}$ and $u = x^2 + 9$, the derivatives are $\frac{dy}{du} = -2u^{-3/2}$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (-2u^{-3/2}) \cdot (2x) \\ &= -2(x^2+9)^{-3/2} \cdot (2x) \\ &= -\frac{4x}{(x^2+9)^{3/2}} \end{aligned}$$

B.3 PRODUCT RULE

B.3.1 DIFFERENTIATING WITH THE PRODUCT RULE

Ex 75: Find the derivative of $f(x) = (x^2 - x)(x^3 + 2)$.

$$f'(x) = \boxed{5x^4 - 4x^3 + 4x - 2}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x^2 - x$ and $v(x) = x^3 + 2$, the derivatives are $u'(x) = 2x - 1$ and $v'(x) = 3x^2$.

$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (2x-1)(x^3+2) + (x^2-x)(3x^2) \\ &= (2x^4 + 4x - x^3 - 2) + (3x^4 - 3x^3) \\ &= 5x^4 - 4x^3 + 4x - 2 \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x^2 - x$ and $v = x^3 + 2$, the derivatives are $\frac{du}{dx} = 2x - 1$ and $\frac{dv}{dx} = 3x^2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (2x-1)(x^3+2) + (x^2-x)(3x^2) \\ &= 2x^4 + 4x - x^3 - 2 + 3x^4 - 3x^3 \\ &= 5x^4 - 4x^3 + 4x - 2 \end{aligned}$$

Ex 76: Find the derivative of $f(x) = (x^2 + 3)(2x - 1)^4$.

$$f'(x) = \boxed{2(2x-1)^3(6x^2-x+12)}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x^2+3$ and $v(x) = (2x-1)^4$, the derivatives are $u'(x) = 2x$ and $v'(x) = 8(2x-1)^3$.

$$\begin{aligned}
f'(x) &= u'(x)v(x) + u(x)v'(x) \\
&= (2x)(2x-1)^4 + (x^2+3)(8(2x-1)^3) \\
&= 2(2x-1)^3[x(2x-1) + 4(x^2+3)] \\
&= 2(2x-1)^3[2x^2 - x + 4x^2 + 12] \\
&= 2(2x-1)^3(6x^2-x+12)
\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x^2+3$ and $v = (2x-1)^4$, the derivatives are $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 8(2x-1)^3$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\
&= (2x)(2x-1)^4 + (x^2+3)(8(2x-1)^3) \\
&= 2(2x-1)^3(6x^2-x+12)
\end{aligned}$$

Ex 77: Find the derivative of $f(x) = \dfrac{1}{x^2}(x^3+1)$.

$$f'(x) = \boxed{1 - \frac{2}{x^3}}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x^{-2}$ and $v(x) = x^3+1$, the derivatives are $u'(x) = -2x^{-3}$ and $v'(x) = 3x^2$.

$$\begin{aligned}
f'(x) &= u'(x)v(x) + u(x)v'(x) \\
&= (-2x^{-3})(x^3+1) + (x^{-2})(3x^2) \\
&= (-2 - 2x^{-3}) + (3) \\
&= 1 - 2x^{-3} \\
&= 1 - \frac{2}{x^3}
\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x^{-2}$ and $v = x^3+1$, the derivatives are $\frac{du}{dx} = -2x^{-3}$ and $\frac{dv}{dx} = 3x^2$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\
&= (-2x^{-3})(x^3+1) + (x^{-2})(3x^2) \\
&= -2 - 2x^{-3} + 3 \\
&= 1 - \frac{2}{x^3}
\end{aligned}$$

Ex 78: Find the derivative of $f(x) = (3x+1)\sqrt{x+1}$.

$$f'(x) = \boxed{\frac{9x+7}{2\sqrt{x+1}}}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = 3x+1$ and $v(x) = \sqrt{x+1}$, the derivatives are $u'(x) = 3$ and $v'(x) = \frac{1}{2\sqrt{x+1}}$.

$$\begin{aligned}
f'(x) &= u'(x)v(x) + u(x)v'(x) \\
&= (3)(\sqrt{x+1}) + (3x+1)\left(\frac{1}{2\sqrt{x+1}}\right) \\
&= \frac{3\sqrt{x+1}\cdot 2\sqrt{x+1} + (3x+1)}{2\sqrt{x+1}} \\
&= \frac{6(x+1) + 3x + 1}{2\sqrt{x+1}} \\
&= \frac{6x + 6 + 3x + 1}{2\sqrt{x+1}} \\
&= \frac{9x+7}{2\sqrt{x+1}}
\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = 3x+1$ and $v = \sqrt{x+1}$, the derivatives are $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = \frac{1}{2\sqrt{x+1}}$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\
&= (3)(\sqrt{x+1}) + (3x+1)\left(\frac{1}{2\sqrt{x+1}}\right) \\
&= \frac{6(x+1) + 3x + 1}{2\sqrt{x+1}} \\
&= \frac{9x+7}{2\sqrt{x+1}}
\end{aligned}$$

B.4 QUOTIENT RULE

B.4.1 DIFFERENTIATING WITH THE QUOTIENT RULE

Ex 79: Find the derivative of $f(x) = \frac{x^2-1}{x^2+1}$.

$$f'(x) = \boxed{\frac{4x}{(x^2+1)^2}}$$

Answer:

- **Using prime notation:**
  For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = x^2-1$ and $v(x) = x^2+1$, the derivatives are $u'(x) = 2x$ and $v'(x) = 2x$.

$$\begin{aligned}
f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\
&= \frac{(2x)(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} \\
&= \frac{(2x^3+2x) - (2x^3-2x)}{(x^2+1)^2} \\
&= \frac{4x}{(x^2+1)^2}
\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \frac{u}{v}$ where $u = x^2-1$ and $v = x^2+1$, the derivatives

are $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}\\ &= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}\\ &= \frac{4x}{(x^2+1)^2}\end{aligned}$$

Ex 80: Find the derivative of $f(x) = \frac{x^2}{x-1}$.

$$f'(x) = \boxed{\frac{x(x-2)}{(x-1)^2}}$$

Answer:

- **Using prime notation:**
For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = x^2$ and $v(x) = x-1$, the derivatives are $u'(x) = 2x$ and $v'(x) = 1$.

$$\begin{aligned}f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}\\ &= \frac{(2x)(x-1) - (x^2)(1)}{(x-1)^2}\\ &= \frac{2x^2-2x-x^2}{(x-1)^2}\\ &= \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}\end{aligned}$$

- **Using Leibniz's Notation** $(y = f(x))$**:**
For $y = \frac{u}{v}$ where $u = x^2$ and $v = x-1$, the derivatives are $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x-1)(2x) - (x^2)(1)}{(x-1)^2}\\ &= \frac{2x^2-2x-x^2}{(x-1)^2}\\ &= \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}\end{aligned}$$

Ex 81: Find the derivative of $f(x) = \frac{\sqrt{x}}{x+1}$.

$$f'(x) = \boxed{\frac{1-x}{2\sqrt{x}(x+1)^2}}$$

Answer:

- **Using prime notation:**
For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = \sqrt{x}$ and $v(x) = x+1$, the derivatives are $u'(x) = \frac{1}{2\sqrt{x}}$ and $v'(x) = 1$.

$$\begin{aligned}f'(x) &= \frac{(\frac{1}{2\sqrt{x}})(x+1) - (\sqrt{x})(1)}{(x+1)^2}\\ &= \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}\\ &= \frac{(x+1) - \sqrt{x}(2\sqrt{x})}{2\sqrt{x}(x+1)^2}\\ &= \frac{x+1-2x}{2\sqrt{x}(x+1)^2}\\ &= \frac{1-x}{2\sqrt{x}(x+1)^2}\end{aligned}$$

- **Using Leibniz's Notation** $(y = f(x))$**:**
For $y = \frac{u}{v}$ where $u = \sqrt{x}$ and $v = x+1$, the derivatives are $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dv}{dx} = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x+1)(\frac{1}{2\sqrt{x}}) - (\sqrt{x})(1)}{(x+1)^2}\\ &= \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2}\end{aligned}$$

Ex 82: Find the derivative of $f(x) = \frac{x+2}{x^2-3}$.

$$f'(x) = \boxed{-\frac{x^2+4x+3}{(x^2-3)^2}}$$

Answer:

- **Using prime notation:**
For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = x+2$ and $v(x) = x^2-3$, the derivatives are $u'(x) = 1$ and $v'(x) = 2x$.

$$\begin{aligned}f'(x) &= \frac{(1)(x^2-3) - (x+2)(2x)}{(x^2-3)^2}\\ &= \frac{x^2-3-(2x^2+4x)}{(x^2-3)^2}\\ &= \frac{x^2-3-2x^2-4x}{(x^2-3)^2}\\ &= \frac{-x^2-4x-3}{(x^2-3)^2} = -\frac{x^2+4x+3}{(x^2-3)^2}\end{aligned}$$

- **Using Leibniz's Notation** $(y = f(x))$**:**
For $y = \frac{u}{v}$ where $u = x+2$ and $v = x^2-3$, the derivatives are $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-3)(1) - (x+2)(2x)}{(x^2-3)^2}\\ &= \frac{x^2-3-2x^2-4x}{(x^2-3)^2}\\ &= -\frac{x^2+4x+3}{(x^2-3)^2}\end{aligned}$$

B.5 IMPLICIT DIFFERENTIATION

B.5.1 FINDING THE DERIVATIVE OF AN IMPLICIT FUNCTION

Ex 83: Find $\frac{dy}{dx}$ for the relation $x^3 + y^3 = 6$.

$$\frac{dy}{dx} = \boxed{-x^2/y^2}$$

Answer: Differentiate both sides with respect to x:

$$\begin{aligned}
\frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6) \\
\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) &= 0 \quad \text{(linearity)} \\
3x^2 + 3y^2 \cdot \frac{dy}{dx} &= 0 \quad \text{(chain rule)} \\
3y^2 \frac{dy}{dx} &= -3x^2 \\
\frac{dy}{dx} &= -\frac{x^2}{y^2} \quad \text{(for } y \neq 0\text{)}
\end{aligned}$$

Ex 84: Find $\frac{dy}{dx}$ for the relation $xy = 4$.

$$\frac{dy}{dx} = \boxed{-y/x}$$

Answer: Differentiate both sides with respect to x, using the product rule on the left side:

$$\frac{d}{dx}(x) \cdot y + x \cdot \frac{d}{dx}(y) = \frac{d}{dx}(4)$$

$$1 \cdot y + x \cdot \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$x\frac{dy}{dx} = -y \implies \frac{dy}{dx} = -\frac{y}{x}$$

Ex 85: Find $\frac{dy}{dx}$ for the relation $x^2 + 3xy - y^2 = 5$.

$$\frac{dy}{dx} = \boxed{\frac{2x+3y}{2y-3x}}$$

Answer: Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$2x + \left(3y + 3x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 0$$

Group the terms with $\frac{dy}{dx}$:

$$(3x - 2y)\frac{dy}{dx} = -2x - 3y$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-2x-3y}{3x-2y} = \frac{2x+3y}{2y-3x}$$

B.5.2 FINDING THE SLOPE OF A TANGENT LINE OF AN IMPLICIT FUNCTION

Ex 86: Find the slope of the tangent to the ellipse $x^2 + 4y^2 = 8$ at the point $(2, 1)$.

$$\text{Slope} = \boxed{-1/2}$$

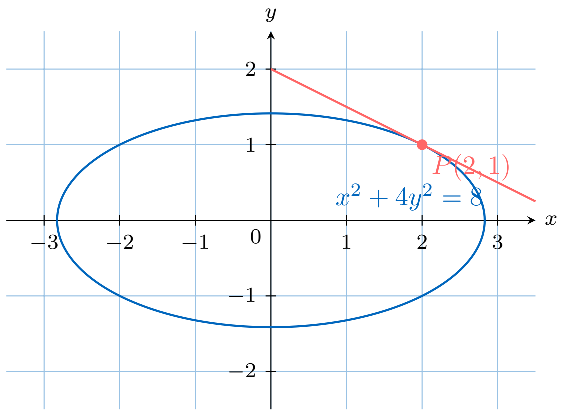
Answer: First, we find the general derivative $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}(x^2 + 4y^2) &= \frac{d}{dx}(8) \\
\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) &= 0 \quad \text{(linearity)} \\
2x + 8y \cdot \frac{dy}{dx} &= 0 \quad \text{(chain rule)} \\
8y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{2x}{8y} = -\frac{x}{4y}
\end{aligned}$$

Now, we evaluate this derivative at the point $(x, y) = (2, 1)$:

$$\frac{dy}{dx} = -\frac{2}{4(1)} = -\frac{1}{2}$$

The slope of the tangent at $(2, 1)$ is $-\frac{1}{2}$.



Ex 87: Find the slope of the tangent to the curve $y^4 - x^3 = 1$ at the point $(2, \sqrt{3})$.

$$\text{Slope} = \boxed{\frac{1}{\sqrt{3}}}$$

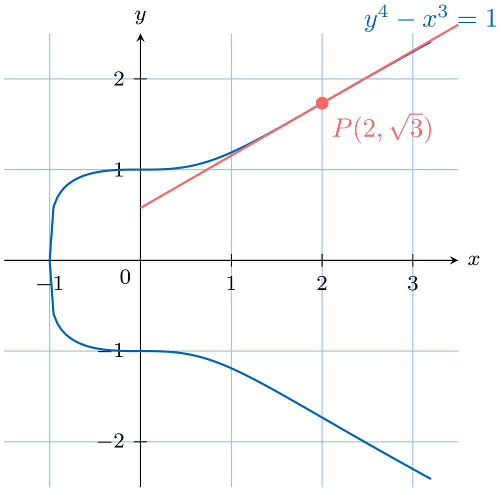
Answer: First, we find the general derivative $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}(y^4 - x^3) &= \frac{d}{dx}(1) \\
\frac{d}{dx}(y^4) - \frac{d}{dx}(x^3) &= 0 \quad \text{(linearity)} \\
4y^3 \cdot \frac{dy}{dx} - 3x^2 &= 0 \quad \text{(chain rule)} \\
4y^3\frac{dy}{dx} &= 3x^2 \\
\frac{dy}{dx} &= \frac{3x^2}{4y^3}
\end{aligned}$$

Now, we evaluate this derivative at the point $(x, y) = (2, \sqrt{3})$:

$$\frac{dy}{dx} = \frac{3(2)^2}{4(\sqrt{3})^3} = \frac{12}{4(3\sqrt{3})} = \frac{12}{12\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The slope of the tangent at $(2, \sqrt{3})$ is $\dfrac{1}{\sqrt{3}}$.



Ex 88: Find the slope of the tangent to the hyperbola $y^2 - x^2 = 3$ at the point $(1, 2)$.

$$\text{Slope} = \boxed{1/2}$$

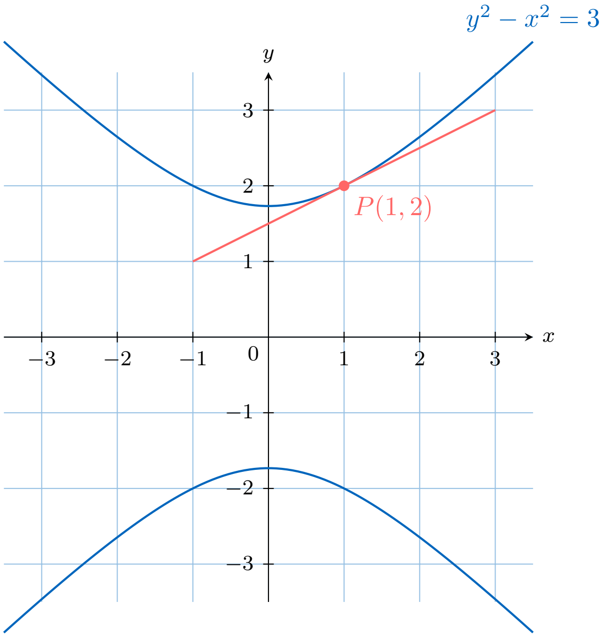
Answer: First, we find the general derivative $\dfrac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}(y^2 - x^2) &= \frac{d}{dx}(3) \\
\frac{d}{dx}(y^2) - \frac{d}{dx}(x^2) &= 0 && \text{(linearity)} \\
2y \cdot \frac{dy}{dx} - 2x &= 0 && \text{(chain rule)} \\
2y\frac{dy}{dx} &= 2x \\
\frac{dy}{dx} &= \frac{2x}{2y} = \frac{x}{y}
\end{aligned}$$

Now, we evaluate this derivative at the point $(x, y) = (1, 2)$:

$$\frac{dy}{dx} = \frac{1}{2}$$

The slope of the tangent at $(1, 2)$ is $\dfrac{1}{2}$.



C DERIVATIVES OF STANDARD FUNCTIONS

C.1 EXPONENTIAL FUNCTIONS

C.1.1 DIFFERENTIATING EXPONENTIAL FUNCTIONS: LEVEL 1

Ex 89: Find the derivative of $f(x) = e^{-x}$.

$$f'(x) = \boxed{-e^{-x}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = e^x$ and the inner function be $u(x) = -x$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = e^x$ and $u'(x) = -1$.

$$\begin{aligned}
f'(x) &= v'(u(x)) \cdot u'(x) \\
&= e^{u(x)} \cdot (-1) \\
&= e^{-x} \cdot (-1) \\
&= -e^{-x}
\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = e^u$ and $u = -x$, the derivatives are $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -1$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= e^u \cdot (-1) \\
&= e^{-x} \cdot (-1) \\
&= -e^{-x}
\end{aligned}$$

Ex 90: Find the derivative of $f(x) = e^{x^2}$.

$$f'(x) = \boxed{2xe^{x^2}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = e^x$ and the inner function be $u(x) = x^2$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = e^x$ and $u'(x) = 2x$.
  $$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= e^{u(x)} \cdot (2x) \\ &= e^{x^2} \cdot (2x) \\ &= 2xe^{x^2} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = e^u$ and $u = x^2$, the derivatives are $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$.
  $$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot (2x) \\ &= e^{x^2} \cdot (2x) \\ &= 2xe^{x^2} \end{aligned}$$

Ex 91: Find the derivative of $f(x) = e^{x^2+2x+2}$.

$$f'(x) = \boxed{(2x+2)e^{x^2+2x+2}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = e^x$ and the inner function be $u(x) = x^2 + 2x + 2$. The derivatives are $v'(x) = e^x$ and $u'(x) = 2x + 2$.
  $$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= e^{x^2+2x+2} \cdot (2x+2) \\ &= (2x+2)e^{x^2+2x+2} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = e^u$ and $u = x^2 + 2x + 2$, the derivatives are $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x + 2$.
  $$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot (2x+2) \\ &= e^{x^2+2x+2} \cdot (2x+2) \\ &= (2x+2)e^{x^2+2x+2} \end{aligned}$$

Ex 92: Find the derivative of $f(x) = e^{2\sqrt{x}}$.

$$f'(x) = \boxed{\frac{e^{2\sqrt{x}}}{\sqrt{x}}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = e^x$ and the inner function be $u(x) = 2\sqrt{x}$. The derivatives are $v'(x) = e^x$ and $u'(x) = \frac{1}{\sqrt{x}}$.
  $$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= e^{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}} \\ &= \frac{e^{2\sqrt{x}}}{\sqrt{x}} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = e^u$ and $u = 2\sqrt{x}$, the derivatives are $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = \frac{1}{\sqrt{x}}$.
  $$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot \frac{1}{\sqrt{x}} \\ &= e^{2\sqrt{x}} \cdot \frac{1}{\sqrt{x}} \\ &= \frac{e^{2\sqrt{x}}}{\sqrt{x}} \end{aligned}$$

C.1.2 DIFFERENTIATING EXPONENTIAL FUNCTIONS: LEVEL 2

Ex 93: Find the derivative of $f(x) = e^x + e^{-x}$.

$$f'(x) = \boxed{e^x - e^{-x}}$$

Answer: Using the sum rule, the derivative of $f(x)$ is the sum of the derivatives of its terms.
The derivative of the first term is $\dfrac{d}{dx}(e^x) = e^x$.
For the second term, e^{-x}, we use the chain rule.

- **Using prime notation:**
  Let the outer function be $v(x) = e^x$ and the inner function be $u(x) = -x$. The derivatives are $v'(x) = e^x$ and $u'(x) = -1$.
  $$\frac{d}{dx}(e^{-x}) = v'(u(x)) \cdot u'(x) = e^{-x} \cdot (-1) = -e^{-x}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  Let $y_2 = e^{-x}$. For $y_2 = e^u$ and $u = -x$, the derivatives are $\frac{dy_2}{du} = e^u$ and $\frac{du}{dx} = -1$.
  $$\frac{dy_2}{dx} = \frac{dy_2}{du} \cdot \frac{du}{dx} = e^u \cdot (-1) = -e^{-x}$$

Combining the derivatives of both terms:

$$f'(x) = e^x - e^{-x}$$

Ex 94: Find the derivative of $f(x) = e^x(x^2 + 1)$.

$$f'(x) = \boxed{e^x(x+1)^2}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = e^x$ and $v(x) = x^2 + 1$, the derivatives are $u'(x) = e^x$ and $v'(x) = 2x$.
  $$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (e^x)(x^2+1) + (e^x)(2x) \\ &= e^x(x^2+1+2x) \\ &= e^x(x^2+2x+1) \\ &= e^x(x+1)^2 \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = uv$ where $u = e^x$ and $v = x^2+1$, the derivatives are $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx}\\ &= (e^x)(x^2+1) + (e^x)(2x)\\ &= e^x(x^2+2x+1)\\ &= e^x(x+1)^2\end{aligned}$$

Ex 95: Find the derivative of $f(x) = \frac{x}{e^x}$.

$$f'(x) = \boxed{\frac{1-x}{e^x}}$$

Answer:

- **Using prime notation:**
For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = x$ and $v(x) = e^x$, the derivatives are $u'(x) = 1$ and $v'(x) = e^x$.

$$\begin{aligned}f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}\\ &= \frac{(1)(e^x) - (x)(e^x)}{(e^x)^2}\\ &= \frac{e^x(1-x)}{e^{2x}}\\ &= \frac{1-x}{e^x}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \frac{u}{v}$ where $u = x$ and $v = e^x$, the derivatives are $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(e^x)(1) - (x)(e^x)}{(e^x)^2}\\ &= \frac{e^x(1-x)}{e^{2x}}\\ &= \frac{1-x}{e^x}\end{aligned}$$

Ex 96: Find the derivative of $f(x) = \sqrt{e^x+1}$.

$$f'(x) = \boxed{\frac{e^x}{2\sqrt{e^x+1}}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \sqrt{x}$ and the inner function be $u(x) = e^x + 1$. The derivatives are $v'(x) = \frac{1}{2\sqrt{x}}$ and $u'(x) = e^x$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x)\\ &= \frac{1}{2\sqrt{u(x)}} \cdot (e^x)\\ &= \frac{1}{2\sqrt{e^x+1}} \cdot e^x\\ &= \frac{e^x}{2\sqrt{e^x+1}}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \sqrt{u}$ and $u = e^x + 1$, the derivatives are $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $\frac{du}{dx} = e^x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{1}{2\sqrt{u}} \cdot (e^x)\\ &= \frac{1}{2\sqrt{e^x+1}} \cdot e^x\\ &= \frac{e^x}{2\sqrt{e^x+1}}\end{aligned}$$

Ex 97: Find the derivative of $f(x) = (1+e^x)^3$.

$$f'(x) = \boxed{3e^x(1+e^x)^2}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = x^3$ and the inner function be $u(x) = 1 + e^x$. The derivatives are $v'(x) = 3x^2$ and $u'(x) = e^x$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x)\\ &= 3(u(x))^2 \cdot e^x\\ &= 3(1+e^x)^2 \cdot e^x\\ &= 3e^x(1+e^x)^2\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = u^3$ and $u = 1+e^x$, the derivatives are $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = e^x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 3u^2 \cdot e^x\\ &= 3(1+e^x)^2 \cdot e^x\\ &= 3e^x(1+e^x)^2\end{aligned}$$

C.2 LOGARITHMIC FUNCTIONS

C.2.1 DIFFERENTIATING LOGARITHMIC FUNCTIONS: LEVEL 1

Ex 98: Find the derivative of $f(x) = \ln(2x)$.

$$f'(x) = \boxed{\frac{1}{x}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \ln(x)$ and the inner function be $u(x) = 2x$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = \frac{1}{x}$ and $u'(x) = 2$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x)\\ &= \frac{1}{u(x)} \cdot (2)\\ &= \frac{1}{2x} \cdot 2\\ &= \frac{1}{x}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \ln(u)$ and $u = 2x$, the derivatives are $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot 2 \\ &= \frac{1}{2x} \cdot 2 \\ &= \frac{1}{x}\end{aligned}$$

Alternatively, using the properties of logarithms, we can rewrite the function as $f(x) = \ln(2) + \ln(x)$. The derivative is then $f'(x) = 0 + \frac{1}{x} = \frac{1}{x}$.

Ex 99: Find the derivative of $f(x) = \ln(x^2 + 3)$.

$$f'(x) = \boxed{\frac{2x}{x^2+3}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \ln(x)$ and the inner function be $u(x) = x^2 + 3$. The derivatives are $v'(x) = \frac{1}{x}$ and $u'(x) = 2x$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{x^2+3} \cdot (2x) \\ &= \frac{2x}{x^2+3}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \ln(u)$ and $u = x^2 + 3$, the derivatives are $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (2x) \\ &= \frac{1}{x^2+3} \cdot (2x) \\ &= \frac{2x}{x^2+3}\end{aligned}$$

Ex 100: Find the derivative of $f(x) = \ln(x^2 + x + 1)$.

$$f'(x) = \boxed{\frac{2x+1}{x^2+x+1}}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \ln(x)$ and the inner function be $u(x) = x^2 + x + 1$. The derivatives are $v'(x) = \frac{1}{x}$ and $u'(x) = 2x + 1$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{x^2+x+1} \cdot (2x+1) \\ &= \frac{2x+1}{x^2+x+1}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \ln(u)$ and $u = x^2 + x + 1$, the derivatives are $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (2x+1) \\ &= \frac{1}{x^2+x+1} \cdot (2x+1) \\ &= \frac{2x+1}{x^2+x+1}\end{aligned}$$

C.2.2 DIFFERENTIATING LOGARITHMIC FUNCTIONS: LEVEL 2

Ex 101: Find the derivative of $f(x) = x^2 \ln(x)$.

$$f'(x) = \boxed{x(1 + 2\ln(x))}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x^2$ and $v(x) = \ln(x)$, the derivatives are $u'(x) = 2x$ and $v'(x) = \frac{1}{x}$.

$$\begin{aligned}f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (2x)(\ln(x)) + (x^2)(\frac{1}{x}) \\ &= 2x\ln(x) + x \\ &= x(2\ln(x) + 1)\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x^2$ and $v = \ln(x)$, the derivatives are $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{x}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (2x)(\ln(x)) + (x^2)(\frac{1}{x}) \\ &= 2x\ln(x) + x \\ &= x(2\ln(x) + 1)\end{aligned}$$

Ex 102: Find the derivative of $f(x) = \frac{\ln(x)}{x^2}$.

$$f'(x) = \boxed{\frac{1 - 2\ln(x)}{x^3}}$$

Answer:

- **Using prime notation:**
  For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = \ln(x)$ and $v(x) = x^2$, the derivatives are $u'(x) = \frac{1}{x}$ and $v'(x) = 2x$.

$$\begin{aligned}f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{(\frac{1}{x})(x^2) - (\ln(x))(2x)}{(x^2)^2} \\ &= \frac{x - 2x\ln(x)}{x^4} \\ &= \frac{x(1 - 2\ln(x))}{x^4} \\ &= \frac{1 - 2\ln(x)}{x^3}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \frac{u}{v}$ where $u = \ln(x)$ and $v = x^2$, the derivatives are $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 2x$.
$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{(x^2)(\frac{1}{x}) - (\ln(x))(2x)}{(x^2)^2}\\ &= \frac{x - 2x\ln(x)}{x^4}\\ &= \frac{1-2\ln(x)}{x^3}\end{aligned}$$

Ex 103: Find the derivative of $f(x) = (\ln(x))^3$.

$$f'(x) = \boxed{\frac{3(\ln(x))^2}{x}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = x^3$ and the inner function be $u(x) = \ln(x)$. The derivatives are $v'(x) = 3x^2$ and $u'(x) = \frac{1}{x}$.
$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x)\\ &= 3(u(x))^2 \cdot \frac{1}{x}\\ &= 3(\ln(x))^2 \cdot \frac{1}{x}\\ &= \frac{3(\ln(x))^2}{x}\end{aligned}$$
- **Using Leibniz's Notation ($y = f(x)$):**
For $y = u^3$ and $u = \ln(x)$, the derivatives are $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = \frac{1}{x}$.
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= (3u^2) \cdot \frac{1}{x}\\ &= 3(\ln(x))^2 \cdot \frac{1}{x}\\ &= \frac{3(\ln(x))^2}{x}\end{aligned}$$

Ex 104: Find the derivative of $f(x) = \ln(\ln(x))$.

$$f'(x) = \boxed{\frac{1}{x\ln(x)}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \ln(x)$ and the inner function be $u(x) = \ln(x)$. The derivatives are $v'(x) = \frac{1}{x}$ and $u'(x) = \frac{1}{x}$.
$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x)\\ &= \frac{1}{u(x)} \cdot \frac{1}{x}\\ &= \frac{1}{\ln(x)} \cdot \frac{1}{x}\\ &= \frac{1}{x\ln(x)}\end{aligned}$$
- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \ln(u)$ and $u = \ln(x)$, the derivatives are $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = \frac{1}{x}$.
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{1}{u} \cdot \frac{1}{x}\\ &= \frac{1}{\ln(x)} \cdot \frac{1}{x}\\ &= \frac{1}{x\ln(x)}\end{aligned}$$

C.2.3 DIFFERENTIATING LOGARITHM FUNCTIONS OF THE FORM $\log_a(x)$

Ex 105: Find the derivative of $f(x) = \log_3(x)$.

$$f'(x) = \boxed{1/(x\ln(3))}$$

Answer: First, we use the change of base formula to rewrite the function in terms of the natural logarithm.

$$f(x) = \log_3(x) = \frac{\ln(x)}{\ln(3)}$$

Now we can differentiate. Note that $\frac{1}{\ln(3)}$ is a constant.

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\frac{\ln(x)}{\ln(3)}\right)\\ &= \frac{1}{\ln(3)} \cdot \frac{d}{dx}(\ln(x))\\ &= \frac{1}{\ln(3)} \cdot \frac{1}{x}\\ &= \frac{1}{x\ln(3)}\end{aligned}$$

Ex 106: Find the derivative of $f(x) = \log_5(x^2+1)$.

$$f'(x) = \boxed{2x/((x^2+1)\ln(5))}$$

Answer: First, use the change of base formula: $f(x) = \dfrac{\ln(x^2+1)}{\ln(5)}$.
Now, differentiate using the chain rule.

$$\begin{aligned}f'(x) &= \frac{1}{\ln(5)} \cdot \frac{d}{dx}(\ln(x^2+1))\\ &= \frac{1}{\ln(5)} \cdot \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1)\\ &= \frac{1}{\ln(5)} \cdot \frac{1}{x^2+1} \cdot (2x)\\ &= \frac{2x}{(x^2+1)\ln(5)}\end{aligned}$$

Ex 107: Find the derivative of $f(x) = x\log_{10}(x)$.

$$f'(x) = \boxed{\log_{10}(x) + 1/\ln(10)}$$

Answer: We use the product rule, $(uv)' = u'v + uv'$. Let $u(x) = x$ and $v(x) = \log_{10}(x)$.
The derivatives of the parts are:

- $u'(x) = 1$

- $v'(x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{x\ln(10)}$

Now, we apply the product rule formula:

$$\begin{aligned} f'(x) &= (1)(\log_{10}(x)) + (x)\left(\frac{1}{x\ln(10)}\right) \\ &= \log_{10}(x) + \frac{1}{\ln(10)} \end{aligned}$$

Note: This can also be written as $\frac{\ln(x)+1}{\ln(10)}$.

Ex 108: Find the derivative of $f(x) = \frac{\log_7(x)}{x}$.

$$f'(x) = \boxed{\frac{1-\ln(x)}{x^2\ln(7)}}$$

Answer: We use the quotient rule, $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$.
Let $u(x) = \log_7(x)$ and $v(x) = x$.
First, find the derivatives of the parts:

- $u'(x) = \frac{d}{dx}(\log_7 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 7}\right) = \frac{1}{x\ln(7)}$
- $v'(x) = \frac{d}{dx}(x) = 1$

Now, apply the quotient rule formula:

$$\begin{aligned} f'(x) &= \frac{(\frac{1}{x\ln(7)})(x) - (\log_7 x)(1)}{x^2} \\ &= \frac{\frac{1}{\ln(7)} - \frac{\ln x}{\ln 7}}{x^2} \\ &= \frac{1-\ln x}{x^2\ln(7)} \end{aligned}$$

C.3 TRIGONOMETRIC FUNCTIONS

C.3.1 DIFFERENTIATING TRIGONOMETRIC FUNCTIONS: LEVEL 1

Ex 109: Find the derivative of $f(x) = \sin(3x)$.

$$f'(x) = \boxed{3\cos(3x)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \sin(x)$ and the inner function be $u(x) = 3x$. We have $f(x) = v(u(x))$. The derivatives are $v'(x) = \cos(x)$ and $u'(x) = 3$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \cos(u(x)) \cdot 3 \\ &= \cos(3x) \cdot 3 \\ &= 3\cos(3x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \sin(u)$ and $u = 3x$, the derivatives are $\frac{dy}{du} = \cos(u)$ and $\frac{du}{dx} = 3$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos(u) \cdot 3 \\ &= \cos(3x) \cdot 3 \\ &= 3\cos(3x) \end{aligned}$$

Ex 110: Find the derivative of $f(x) = \cos(x^2)$.

$$f'(x) = \boxed{-2x\sin(x^2)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \cos(x)$ and the inner function be $u(x) = x^2$. The derivatives are $v'(x) = -\sin(x)$ and $u'(x) = 2x$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= -\sin(u(x)) \cdot (2x) \\ &= -\sin(x^2) \cdot (2x) \\ &= -2x\sin(x^2) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \cos(u)$ and $u = x^2$, the derivatives are $\frac{dy}{du} = -\sin(u)$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\sin(u) \cdot (2x) \\ &= -\sin(x^2) \cdot (2x) \\ &= -2x\sin(x^2) \end{aligned}$$

Ex 111: Find the derivative of $f(x) = \tan(2x+1)$.

$$f'(x) = \boxed{2\sec^2(2x+1)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \tan(x)$ and the inner function be $u(x) = 2x+1$. The derivatives are $v'(x) = \sec^2(x)$ and $u'(x) = 2$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \sec^2(u(x)) \cdot 2 \\ &= \sec^2(2x+1) \cdot 2 \\ &= 2\sec^2(2x+1) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \tan(u)$ and $u = 2x+1$, the derivatives are $\frac{dy}{du} = \sec^2(u)$ and $\frac{du}{dx} = 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \sec^2(u) \cdot 2 \\ &= \sec^2(2x+1) \cdot 2 \\ &= 2\sec^2(2x+1) \end{aligned}$$

Note that from the trigonometric identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, this result is equivalent to $2(1+\tan^2(2x+1))$.

C.3.2 DIFFERENTIATING TRIGONOMETRIC FUNCTIONS: LEVEL 2

Ex 112: Find the derivative of $f(x) = x\cos(x)$.

$$f'(x) = \boxed{\cos(x) - x\sin(x)}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x$ and $v(x) = \cos(x)$, the derivatives are $u'(x) = 1$ and $v'(x) = -\sin(x)$.
$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (1)(\cos(x)) + (x)(-\sin(x)) \\ &= \cos(x) - x\sin(x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x$ and $v = \cos(x)$, the derivatives are $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\sin(x)$.
$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (1)(\cos(x)) + (x)(-\sin(x)) \\ &= \cos(x) - x\sin(x) \end{aligned}$$

Ex 113: Find the derivative of $f(x) = \sin^3(x)$.
$$f'(x) = \boxed{3\sin^2(x)\cos(x)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = x^3$ and the inner function be $u(x) = \sin(x)$. The derivatives are $v'(x) = 3x^2$ and $u'(x) = \cos(x)$.
$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= 3(u(x))^2 \cdot \cos(x) \\ &= 3(\sin(x))^2 \cdot \cos(x) \\ &= 3\sin^2(x)\cos(x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = u^3$ and $u = \sin(x)$, the derivatives are $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = \cos(x)$.
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (3u^2) \cdot \cos(x) \\ &= 3(\sin(x))^2 \cdot \cos(x) \\ &= 3\sin^2(x)\cos(x) \end{aligned}$$

Ex 114: Find the derivative of $f(x) = \frac{\sin(x)}{x}$.
$$f'(x) = \boxed{\frac{x\cos(x) - \sin(x)}{x^2}}$$

Answer:

- **Using prime notation:**
  For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = \sin(x)$ and $v(x) = x$, the derivatives are $u'(x) = \cos(x)$ and $v'(x) = 1$.
$$\begin{aligned} f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{(\cos(x))(x) - (\sin(x))(1)}{x^2} \\ &= \frac{x\cos(x) - \sin(x)}{x^2} \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \frac{u}{v}$ where $u = \sin(x)$ and $v = x$, the derivatives are $\frac{du}{dx} = \cos(x)$ and $\frac{dv}{dx} = 1$.
$$\begin{aligned} \frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(x)(\cos(x)) - (\sin(x))(1)}{x^2} \\ &= \frac{x\cos(x) - \sin(x)}{x^2} \end{aligned}$$

Ex 115: Find the derivative of $f(x) = e^x\sin(x)$.
$$f'(x) = \boxed{e^x(\sin(x) + \cos(x))}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = e^x$ and $v(x) = \sin(x)$, the derivatives are $u'(x) = e^x$ and $v'(x) = \cos(x)$.
$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (e^x)(\sin(x)) + (e^x)(\cos(x)) \\ &= e^x(\sin(x) + \cos(x)) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = e^x$ and $v = \sin(x)$, the derivatives are $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \cos(x)$.
$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (e^x)(\sin(x)) + (e^x)(\cos(x)) \\ &= e^x(\sin(x) + \cos(x)) \end{aligned}$$

C.3.3 FINDING THE SLOPE OF A TANGENT LINE OF AN IMPLICIT FUNCTION

Ex 116: A curve is defined by the implicit equation $x^3 + \sin(y) = xy$.

1. Show that $\frac{dy}{dx} = \frac{y - 3x^2}{\cos(y) - x}$.
2. Find the slope of the tangent to the curve at the point $(0, \pi)$.

Answer:

1. We differentiate both sides with respect to x. We must use the chain rule for $\sin(y)$ and the product rule for xy.
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(\sin y) = \frac{d}{dx}(xy)$$
$$3x^2 + \cos(y) \cdot \frac{dy}{dx} = \left(\frac{d}{dx}(x) \cdot y + x \cdot \frac{d}{dx}(y)\right)$$
$$3x^2 + \cos(y)\frac{dy}{dx} = 1 \cdot y + x\frac{dy}{dx}$$
   Now, we group the terms with $\frac{dy}{dx}$ and solve.
$$\cos(y)\frac{dy}{dx} - x\frac{dy}{dx} = y - 3x^2$$
$$\frac{dy}{dx}(\cos y - x) = y - 3x^2$$
$$\frac{dy}{dx} = \frac{y - 3x^2}{\cos y - x}$$

2. We evaluate the derivative at $(x, y) = (0, \pi)$.

$$\left.\frac{dy}{dx}\right|_{(0,\pi)} = \frac{\pi - 3(0)^2}{\cos(\pi) - 0} = \frac{\pi}{-1} = -\pi$$

The slope of the tangent at $(0, \pi)$ is $-\pi$.

C.3.4 DIFFERENTIATING OTHER TRIGONOMETRIC FUNCTIONS: LEVEL 1

Ex 117: Find the derivative of $f(x) = \sec(4x)$.

$$f'(x) = \boxed{4\sec(4x)\tan(4x)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \sec(x)$ and the inner function be $u(x) = 4x$. The derivatives are $v'(x) = \sec(x)\tan(x)$ and $u'(x) = 4$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \sec(u(x))\tan(u(x)) \cdot 4 \\ &= 4\sec(4x)\tan(4x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \sec(u)$ and $u = 4x$, the derivatives are $\frac{dy}{du} = \sec(u)\tan(u)$ and $\frac{du}{dx} = 4$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \sec(u)\tan(u) \cdot 4 \\ &= 4\sec(4x)\tan(4x) \end{aligned}$$

Ex 118: Find the derivative of $f(x) = \cot(x^2)$.

$$f'(x) = \boxed{-2x\csc^2(x^2)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \cot(x)$ and the inner function be $u(x) = x^2$. The derivatives are $v'(x) = -\csc^2(x)$ and $u'(x) = 2x$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= -\csc^2(u(x)) \cdot (2x) \\ &= -2x\csc^2(x^2) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \cot(u)$ and $u = x^2$, the derivatives are $\frac{dy}{du} = -\csc^2(u)$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\csc^2(u) \cdot (2x) \\ &= -2x\csc^2(x^2) \end{aligned}$$

Ex 119: Find the derivative of $f(x) = \csc(2x + 1)$.

$$f'(x) = \boxed{-2\csc(2x+1)\cot(2x+1)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = \csc(x)$ and the inner function be $u(x) = 2x + 1$. The derivatives are $v'(x) = -\csc(x)\cot(x)$ and $u'(x) = 2$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= -\csc(u(x))\cot(u(x)) \cdot 2 \\ &= -2\csc(2x+1)\cot(2x+1) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = \csc(u)$ and $u = 2x + 1$, the derivatives are $\frac{dy}{du} = -\csc(u)\cot(u)$ and $\frac{du}{dx} = 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\csc(u)\cot(u) \cdot 2 \\ &= -2\csc(2x+1)\cot(2x+1) \end{aligned}$$

C.3.5 DIFFERENTIATING OTHER TRIGONOMETRIC FUNCTIONS: LEVEL 2

Ex 120: Find the derivative of $f(x) = x\sec(x)$.

$$f'(x) = \boxed{\sec(x) + x\sec(x)\tan(x)}$$

Answer:

- **Using prime notation:**
  For $f(x) = u(x)v(x)$ where $u(x) = x$ and $v(x) = \sec(x)$, the derivatives are $u'(x) = 1$ and $v'(x) = \sec(x)\tan(x)$.

$$\begin{aligned} f'(x) &= u'(x)v(x) + u(x)v'(x) \\ &= (1)(\sec(x)) + (x)(\sec(x)\tan(x)) \\ &= \sec(x) + x\sec(x)\tan(x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
  For $y = uv$ where $u = x$ and $v = \sec(x)$, the derivatives are $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \sec(x)\tan(x)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (1)(\sec(x)) + (x)(\sec(x)\tan(x)) \\ &= \sec(x) + x\sec(x)\tan(x) \end{aligned}$$

Ex 121: Find the derivative of $f(x) = \cot^2(x)$.

$$f'(x) = \boxed{-2\cot(x)\csc^2(x)}$$

Answer:

- **Using prime notation:**
  Let the outer function be $v(x) = x^2$ and the inner function be $u(x) = \cot(x)$. The derivatives are $v'(x) = 2x$ and $u'(x) = -\csc^2(x)$.

$$\begin{aligned} f'(x) &= v'(u(x)) \cdot u'(x) \\ &= 2(u(x)) \cdot (-\csc^2(x)) \\ &= 2\cot(x)(-\csc^2(x)) \\ &= -2\cot(x)\csc^2(x) \end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = u^2$ and $u = \cot(x)$, the derivatives are $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = -\csc^2(x)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (2u) \cdot (-\csc^2(x)) \\ &= 2\cot(x)(-\csc^2(x)) \\ &= -2\cot(x)\csc^2(x)\end{aligned}$$

Ex 122: Find the derivative of $f(x) = \frac{\csc(x)}{x}$.

$$f'(x) = \boxed{\frac{-x\csc(x)\cot(x) - \csc(x)}{x^2}}$$

Answer:

- **Using prime notation:**
For $f(x) = \frac{u(x)}{v(x)}$ where $u(x) = \csc(x)$ and $v(x) = x$, the derivatives are $u'(x) = -\csc(x)\cot(x)$ and $v'(x) = 1$.

$$\begin{aligned}f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{(-\csc(x)\cot(x))(x) - (\csc(x))(1)}{x^2} \\ &= \frac{-x\csc(x)\cot(x) - \csc(x)}{x^2}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \frac{u}{v}$ where $u = \csc(x)$ and $v = x$, the derivatives are $\frac{du}{dx} = -\csc(x)\cot(x)$ and $\frac{dv}{dx} = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(x)(-\csc(x)\cot(x)) - (\csc(x))(1)}{x^2} \\ &= \frac{-x\csc(x)\cot(x) - \csc(x)}{x^2}\end{aligned}$$

C.4 INVERSE TRIGONOMETRIC FUNCTIONS

C.4.1 DIFFERENTIATING INVERSE TRIGONOMETRIC FUNCTIONS: LEVEL 1

Ex 123: Find the derivative of $f(x) = \arcsin(2x)$.

$$f'(x) = \boxed{2/\sqrt{1-4x^2}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \arcsin(x)$ and the inner function be $u(x) = 2x$. The derivatives are $v'(x) = \dfrac{1}{\sqrt{1-x^2}}$ and $u'(x) = 2$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{\sqrt{1-(u(x))^2}} \cdot 2 \\ &= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-4x^2}}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \arcsin(u)$ and $u = 2x$, the derivatives are $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx} = 2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{\sqrt{1-u^2}} \cdot 2 \\ &= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-4x^2}}\end{aligned}$$

Ex 124: Find the derivative of $f(x) = \arctan(x^3)$.

$$f'(x) = \boxed{3x^2/(1+x^6)}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \arctan(x)$ and the inner function be $u(x) = x^3$. The derivatives are $v'(x) = \dfrac{1}{1+x^2}$ and $u'(x) = 3x^2$.

$$\begin{aligned}f'(x) &= v'(u(x)) \cdot u'(x) \\ &= \frac{1}{1+(u(x))^2} \cdot (3x^2) \\ &= \frac{1}{1+(x^3)^2} \cdot (3x^2) \\ &= \frac{3x^2}{1+x^6}\end{aligned}$$

- **Using Leibniz's Notation ($y = f(x)$):**
For $y = \arctan(u)$ and $u = x^3$, the derivatives are $\dfrac{dy}{du} = \dfrac{1}{1+u^2}$ and $\dfrac{du}{dx} = 3x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{1+u^2} \cdot (3x^2) \\ &= \frac{1}{1+(x^3)^2} \cdot (3x^2) \\ &= \frac{3x^2}{1+x^6}\end{aligned}$$

Ex 125: Find the derivative of $f(x) = \arccos(x+1)$.

$$f'(x) = \boxed{-\frac{1}{\sqrt{-x^2-2x}}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x) = \arccos(x)$ and the inner function be $u(x) = x+1$. The derivatives are $v'(x) =$

$-\dfrac{1}{\sqrt{1-x^2}}$ and $u'(x)=1$.

$$\begin{aligned} f'(x) &= v'(u(x))\cdot u'(x) \\ &= -\frac{1}{\sqrt{1-(u(x))^2}}\cdot 1 \\ &= -\frac{1}{\sqrt{1-(x+1)^2}} \\ &= -\frac{1}{\sqrt{1-(x^2+2x+1)}} \\ &= -\frac{1}{\sqrt{-x^2-2x}} \end{aligned}$$

- **Using Leibniz's Notation ($y=f(x)$):**
For $y=\arccos(u)$ and $u=x+1$, the derivatives are $\dfrac{dy}{du}=-\dfrac{1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx}=1$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= -\frac{1}{\sqrt{1-u^2}}\cdot 1 \\ &= -\frac{1}{\sqrt{1-(x+1)^2}} \\ &= -\frac{1}{\sqrt{-x^2-2x}} \end{aligned}$$

C.4.2 DIFFERENTIATING INVERSE TRIGONOMETRIC FUNCTIONS: LEVEL 2

Ex 126: Find the derivative of $f(x)=x\arctan(x)$.

$$f'(x)=\boxed{\arctan(x)+x/(1+x^2)}$$

Answer:

- **Using prime notation:**
For $f(x)=u(x)v(x)$ where $u(x)=x$ and $v(x)=\arctan(x)$, the derivatives are $u'(x)=1$ and $v'(x)=\dfrac{1}{1+x^2}$.

$$\begin{aligned} f'(x) &= u'(x)v(x)+u(x)v'(x) \\ &= (1)(\arctan(x))+(x)\left(\frac{1}{1+x^2}\right) \\ &= \arctan(x)+\frac{x}{1+x^2} \end{aligned}$$

- **Using Leibniz's Notation ($y=f(x)$):**
For $y=uv$ where $u=x$ and $v=\arctan(x)$, the derivatives are $\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=\dfrac{1}{1+x^2}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v+u\frac{dv}{dx} \\ &= (1)(\arctan(x))+(x)\left(\frac{1}{1+x^2}\right) \\ &= \arctan(x)+\frac{x}{1+x^2} \end{aligned}$$

Ex 127: Find the derivative of $f(x)=(\arcsin x)^3$.

$$f'(x)=\boxed{3(\arcsin x)^2/\sqrt{1-x^2}}$$

Answer:

- **Using prime notation:**
Let the outer function be $v(x)=x^3$ and the inner function be $u(x)=\arcsin(x)$. The derivatives are $v'(x)=3x^2$ and $u'(x)=\dfrac{1}{\sqrt{1-x^2}}$.

$$\begin{aligned} f'(x) &= v'(u(x))\cdot u'(x) \\ &= 3(u(x))^2\cdot\frac{1}{\sqrt{1-x^2}} \\ &= 3(\arcsin x)^2\cdot\frac{1}{\sqrt{1-x^2}} \\ &= \frac{3(\arcsin x)^2}{\sqrt{1-x^2}} \end{aligned}$$

- **Using Leibniz's Notation ($y=f(x)$):**
For $y=u^3$ and $u=\arcsin(x)$, the derivatives are $\dfrac{dy}{du}=3u^2$ and $\dfrac{du}{dx}=\dfrac{1}{\sqrt{1-x^2}}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= 3u^2\cdot\frac{1}{\sqrt{1-x^2}} \\ &= 3(\arcsin x)^2\cdot\frac{1}{\sqrt{1-x^2}} \\ &= \frac{3(\arcsin x)^2}{\sqrt{1-x^2}} \end{aligned}$$

Ex 128: Find the derivative of $f(x)=\dfrac{\arccos x}{x}$.

$$f'(x)=\boxed{\frac{-x-\sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}}}$$

Answer:

- **Using prime notation:**
For $f(x)=\frac{u(x)}{v(x)}$ where $u(x)=\arccos(x)$ and $v(x)=x$, the derivatives are $u'(x)=-\dfrac{1}{\sqrt{1-x^2}}$ and $v'(x)=1$.

$$\begin{aligned} f'(x) &= \frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2} \\ &= \frac{\left(-\frac{1}{\sqrt{1-x^2}}\right)(x)-(\arccos x)(1)}{x^2} \\ &= \frac{-\frac{x}{\sqrt{1-x^2}}-\arccos x}{x^2} \\ &= \frac{\frac{-x-\sqrt{1-x^2}\arccos x}{\sqrt{1-x^2}}}{x^2} \\ &= \frac{-x-\sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}} \end{aligned}$$

- **Using Leibniz's Notation ($y=f(x)$):**
For $y=\frac{u}{v}$ where $u=\arccos(x)$ and $v=x$, the derivatives

are $\dfrac{du}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{dv}{dx} = 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(x)\left(-\frac{1}{\sqrt{1-x^2}}\right) - (\arccos x)(1)}{x^2} \\ &= \frac{-\frac{x}{\sqrt{1-x^2}} - \arccos x}{x^2} \\ &= \frac{-x - \sqrt{1-x^2}\arccos x}{x^2\sqrt{1-x^2}}\end{aligned}$$

D SECOND DERIVATIVE

D.1 DEFINITION

D.1.1 CALCULATING THE FIRST AND SECOND DERIVATIVE: LEVEL 1

Ex 129: Find the first and second derivatives of $f(x) = x^4 - 3x^2 + 7$.

$$\begin{aligned}f'(x) &= \boxed{4x^3 - 6x} \\ f''(x) &= \boxed{12x^2 - 6}\end{aligned}$$

Answer: First, we find the first derivative, $f'(x)$, by applying the power rule to each term.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^4 - 3x^2 + 7) \\ &= 4x^3 - 3(2x) + 0 \\ &= 4x^3 - 6x\end{aligned}$$

Next, we find the second derivative, $f''(x)$, by differentiating $f'(x)$.

$$\begin{aligned}f''(x) &= \frac{d}{dx}(4x^3 - 6x) \\ &= 4(3x^2) - 6 \\ &= 12x^2 - 6\end{aligned}$$

Ex 130: Find the first and second derivatives of $f(x) = e^{5x}$.

$$\begin{aligned}f'(x) &= \boxed{5e^{5x}} \\ f''(x) &= \boxed{25e^{5x}}\end{aligned}$$

Answer: First, we find the first derivative, $f'(x)$, using the chain rule.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(e^{5x}) \\ &= e^{5x} \cdot \frac{d}{dx}(5x) \\ &= 5e^{5x}\end{aligned}$$

Next, we find the second derivative, $f''(x)$, by differentiating $f'(x)$.

$$\begin{aligned}f''(x) &= \frac{d}{dx}(5e^{5x}) \\ &= 5 \cdot \frac{d}{dx}(e^{5x}) \\ &= 5 \cdot (5e^{5x}) \\ &= 25e^{5x}\end{aligned}$$

Ex 131: Find the first and second derivatives of $f(x) = \sin(2x)$.

$$\begin{aligned}f'(x) &= \boxed{2\cos(2x)} \\ f''(x) &= \boxed{-4\sin(2x)}\end{aligned}$$

Answer: First, we find the first derivative, $f'(x)$, using the chain rule.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(\sin(2x)) \\ &= \cos(2x) \cdot \frac{d}{dx}(2x) \\ &= 2\cos(2x)\end{aligned}$$

Next, we find the second derivative, $f''(x)$, by differentiating $f'(x)$.

$$\begin{aligned}f''(x) &= \frac{d}{dx}(2\cos(2x)) \\ &= 2 \cdot (-\sin(2x)) \cdot \frac{d}{dx}(2x) \\ &= -2\sin(2x) \cdot 2 \\ &= -4\sin(2x)\end{aligned}$$

D.1.2 CALCULATING THE FIRST AND SECOND DERIVATIVE: LEVEL 2

Ex 132: Find the first and second derivatives of $f(x) = x^2\ln(x)$.

$$\begin{aligned}f'(x) &= \boxed{x(1 + 2\ln(x))} \\ f''(x) &= \boxed{3 + 2\ln(x)}\end{aligned}$$

Answer: To find the first derivative, $f'(x)$, we use the product rule.

$$\begin{aligned}f'(x) &= \frac{d}{dx}(x^2) \cdot \ln(x) + x^2 \cdot \frac{d}{dx}(\ln x) \\ &= 2x\ln(x) + x^2 \cdot \frac{1}{x} \\ &= 2x\ln(x) + x \\ &= x(2\ln(x) + 1)\end{aligned}$$

To find the second derivative, $f''(x)$, we differentiate $f'(x) = 2x\ln(x) + x$, again using the product rule for the first term.

$$\begin{aligned}f''(x) &= \frac{d}{dx}(2x\ln(x) + x) \\ &= \left[\frac{d}{dx}(2x) \cdot \ln(x) + 2x \cdot \frac{d}{dx}(\ln x)\right] + \frac{d}{dx}(x) \\ &= \left[2\ln(x) + 2x \cdot \frac{1}{x}\right] + 1 \\ &= 2\ln(x) + 2 + 1 \\ &= 2\ln(x) + 3\end{aligned}$$

Ex 133: Find the first and second derivatives of $f(x) = \frac{x}{x+1}$.

$$\begin{aligned}f'(x) &= \boxed{\frac{1}{(x+1)^2}} \\ f''(x) &= \boxed{-\frac{2}{(x+1)^3}}\end{aligned}$$

Answer: To find the first derivative, $f'(x)$, we use the quotient rule.

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(x)\cdot(x+1) - x\cdot\frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} \\ &= \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} \end{aligned}$$

To find the second derivative, $f''(x)$, we can rewrite $f'(x) = (x+1)^{-2}$ and use the chain rule.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left((x+1)^{-2}\right) \\ &= -2(x+1)^{-3}\cdot\frac{d}{dx}(x+1) \\ &= -2(x+1)^{-3}\cdot 1 \\ &= -\frac{2}{(x+1)^3} \end{aligned}$$

Ex 134: Find the first and second derivatives of $f(x) = e^x\cos(x)$.

$$\begin{aligned} f'(x) &= \boxed{e^x(\cos(x)-\sin(x))} \\ f''(x) &= \boxed{-2e^x\sin(x)} \end{aligned}$$

Answer: To find the first derivative, $f'(x)$, we use the product rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^x)\cdot\cos(x) + e^x\cdot\frac{d}{dx}(\cos x) \\ &= e^x\cos(x) + e^x(-\sin x) \\ &= e^x(\cos(x)-\sin(x)) \end{aligned}$$

To find the second derivative, $f''(x)$, we differentiate $f'(x)$ using the product rule again.

$$\begin{aligned} f''(x) &= \frac{d}{dx}(e^x)\cdot(\cos(x)-\sin(x)) + e^x\cdot\frac{d}{dx}(\cos(x)-\sin(x)) \\ &= e^x(\cos(x)-\sin(x)) + e^x(-\sin(x)-\cos(x)) \\ &= e^x\cos(x) - e^x\sin(x) - e^x\sin(x) - e^x\cos(x) \\ &= -2e^x\sin(x) \end{aligned}$$

E FINDING LIMITS OF INDETERMINATE FORMS

E.1 L'HÔPITAL'S RULE

E.1.1 APPLYING L'HÔPITAL'S RULE: LEVEL 1

Ex 135: Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Answer: First, check the form by substitution:

$$\lim_{x\to 0}\sin x = 0, \quad \lim_{x\to 0} x = 0.$$

This is the indeterminate form $\dfrac{0}{0}$.
By L'Hôpital's Rule, we consider the ratio of derivatives:

$$\begin{aligned} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} &= \frac{\cos x}{1} \\ &\xrightarrow[x\to 0]{} \frac{\cos(0)}{1} = 1 \end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Ex 136: Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

Answer: Substitute $x = 0$:

$$\lim_{x\to 0}\ln(1+x) = 0, \quad \lim_{x\to 0} x = 0.$$

This gives the indeterminate form $\dfrac{0}{0}$.
By L'Hôpital's Rule, consider the ratio of derivatives:

$$\begin{aligned} \frac{\frac{d}{dx}(\ln(1+x))}{\frac{d}{dx}(x)} &= \frac{\frac{1}{1+x}}{1} \\ &\xrightarrow[x\to 0]{} \frac{1}{1+0} = 1 \end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$

Ex 137: Evaluate the limit $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}$.

Answer: Substitute $x = 0$:

$$\lim_{x\to 0}(e^x-1-x) = 0, \quad \lim_{x\to 0} x^2 = 0.$$

This gives the indeterminate form $\dfrac{0}{0}$.
Apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{e^x-1}{2x}.$$

Again, substituting $x = 0$ gives $\dfrac{0}{0}$. Apply L'Hôpital's Rule again:

$$\begin{aligned} \frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(2x)} &= \frac{e^x}{2} \\ &\xrightarrow[x\to 0]{} \frac{1}{2} \end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \tfrac{1}{2}.$$

Ex 138: Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$.

Answer: Substitute as $x\to\infty$:

$$\lim_{x\to\infty}\ln x = \infty, \quad \lim_{x\to\infty} x = \infty.$$

This is the indeterminate form $\dfrac{\infty}{\infty}$.
By L'Hôpital's Rule:

$$\begin{aligned} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} &= \frac{\frac{1}{x}}{1} \\ &\xrightarrow[x\to\infty]{} 0 \end{aligned}$$

Therefore,

$$\lim_{x\to\infty}\frac{\ln x}{x} = 0.$$

Ex 139: Evaluate the limit $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$.

Answer: Substitute as $x \to \infty$:

$$\lim_{x\to\infty} x = \infty, \quad \lim_{x\to\infty} e^x = \infty.$$

This is the indeterminate form $\dfrac{\infty}{\infty}$.
By L'Hôpital's Rule:

$$\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)} = \frac{1}{e^x} \xrightarrow[x\to\infty]{} 0$$

Therefore,

$$\lim_{x\to\infty} \frac{x}{e^x} = 0.$$

E.1.2 APPLYING L'HÔPITAL'S RULE: LEVEL 2

Ex 140: Consider the function $f(x) = \dfrac{\ln(1+x)}{x}$ for $x > 0$.

1. Show that $\displaystyle\lim_{x\to 0^+} \frac{\ln(1+x)}{x}$ exists and find its value.
2. Hence, determine $\displaystyle\lim_{x\to 0^+} \frac{\ln(1+2x)}{x}$.

Answer:

1. **Limit of $\frac{\ln(1+x)}{x}$ as $x \to 0^+$**
Substitution as $x \to 0^+$:
$$\lim_{x\to 0^+} \ln(1+x) = \ln(1) = 0, \quad \lim_{x\to 0^+} x = 0.$$
This is the indeterminate form $\dfrac{0}{0}$.
By L'Hôpital's Rule:
$$\frac{\frac{d}{dx}(\ln(1+x))}{\frac{d}{dx}(x)} = \frac{\frac{1}{1+x}}{1} \xrightarrow[x\to 0^+]{} 1$$
Therefore,
$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 1.$$
2. **Limit of $\frac{\ln(1+2x)}{x}$ as $x \to 0^+$**
Let $u = 2x \implies u \to 0^+$ as $x \to 0^+$. Then
$$\lim_{x\to 0^+} \frac{\ln(1+2x)}{x} = \lim_{u\to 0^+} \frac{\ln(1+u)}{u/2} = 2 \lim_{u\to 0^+} \frac{\ln(1+u)}{u}.$$
From part (1), this limit equals $2(1) = 2$.
Therefore,
$$\lim_{x\to 0^+} \frac{\ln(1+2x)}{x} = 2.$$

Ex 141:

1. Prove that $\displaystyle\lim_{x\to\infty} x\ln\left(1+\tfrac{1}{x}\right) = 1$.
2. By writing $\left(1+\frac{1}{x}\right)^x = e^{x\ln(1+\frac{1}{x})}$ and using the fact that $f(x) = e^x$ is continuous on $\mathbb{R}$, prove that
$$\lim_{x\to\infty} \left(1+\tfrac{1}{x}\right)^x = e.$$

Answer:

1. **Limit of $x\ln(1+1/x)$**
Substitute as $x \to \infty$:
$$\lim_{x\to\infty} \ln(1+1/x) = 0, \quad \lim_{x\to\infty} x = \infty.$$
This is the indeterminate form $0 \cdot \infty$.
Rewrite as a quotient:
$$x\ln\left(1+\tfrac{1}{x}\right) = \frac{\ln(1+1/x)}{1/x}.$$
By L'Hôpital's Rule:
$$\begin{aligned}\frac{\frac{d}{dx}\left(\ln(1+\frac{1}{x})\right)}{\frac{d}{dx}(\frac{1}{x})} &= \frac{\frac{\frac{d}{dx}(1+\frac{1}{x})}{(1+\frac{1}{x})}}{-1/x^2} \\ &= \frac{\frac{-1/x^2}{(1+\frac{1}{x})}}{-1/x^2} \\ &= \frac{1}{1+\frac{1}{x}} \\ &\xrightarrow[x\to\infty]{} \frac{1}{1+0} = 1\end{aligned}$$
Therefore,
$$\lim_{x\to\infty} x\ln\left(1+\tfrac{1}{x}\right) = 1.$$
2. **Limit of $\left(1+\frac{1}{x}\right)^x$**
$$\begin{aligned}\lim_{x\to\infty} \left(1+\tfrac{1}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left[\left(1+\frac{1}{x}\right)^x\right]} \\ &= \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} \\ &= e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} \quad (e^x \text{ is continious}) \\ &= e^1 \\ &= e\end{aligned}$$

Ex 142: On considère l'expression x^x pour $x > 0$.

1. En posant $u = \frac{1}{x}$ (donc $x = \frac{1}{u}$), réécrire x^x sous la forme
$$x^x = e^{-\frac{\ln u}{u}}.$$
2. En déduire, à l'aide de cette réécriture et de la règle de l'Hôpital, la valeur de la limite
$$\lim_{x\to 0^+} x^x.$$

Answer:

1. **Rewriting x^x**
$$\begin{aligned}x^x &= e^{\ln(x^x)} \\ &= e^{x\ln x} \\ \text{Let } u = \tfrac{1}{x} &\implies x = \tfrac{1}{u}, \\ x\ln x &= \tfrac{1}{u}\ln\left(\tfrac{1}{u}\right) \\ &= \tfrac{1}{u}(-\ln u) \\ &= -\tfrac{\ln u}{u}.\end{aligned}$$
Therefore,
$$x^x = e^{-\frac{\ln u}{u}}.$$

2. **Limit as $x \to 0^+$**
   As $x \to 0^+$, we have $u = \frac{1}{x} \to \infty$. Then

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{u\to\infty} e^{-\frac{\ln u}{u}} \\ &= e^{\lim\limits_{u\to\infty} -\frac{\ln u}{u}} \quad (e^x \text{ is continuous}).\end{aligned}$$

   Now check the exponent:

$$\begin{aligned}\lim_{u\to\infty} \frac{\ln u}{u} &= \lim_{u\to\infty} \frac{\frac{1}{u}}{1} \quad \text{(L'Hôpital's Rule)} \\ &= 0.\end{aligned}$$

   Therefore,

$$\lim_{x\to 0^+} x^x = e^0 = 1.$$