DISCRETE RANDOM VARIABLES

A RANDOM VARIABLES

A.1 DEFINITIONS

A.1.1 FINDING THE VALUE OF A RANDOM VARIABLE

Ex 1: Let X represent the number of heads obtained when tossing two fair coins: a red coin and a blue coin. Determine the value of X for each of the following outcomes:

- $X(T,T) = \boxed{0}$,
- $X(H,T) = \boxed{1}$,
- $X(T,H) = \boxed{1}$,
- $X(H,H) = \boxed{2}$.

Answer: For each outcome, X counts the number of heads (H) from the two coins:

- $X(T,T)$: The red coin shows tails (T) and the blue coin shows tails (T). There are 0 heads, so $X(T,T) = 0$.
- $X(H,T)$: The red coin shows heads (H) and the blue coin shows tails (T). There is 1 head, so $X(H,T) = 1$.
- $X(T,H)$: The red coin shows tails (T) and the blue coin shows heads (H). There is 1 head, so $X(T,H) = 1$.
- $X(H,H)$: The red coin shows heads (H) and the blue coin shows heads (H). There are 2 heads, so $X(H,H) = 2$.

Ex 2: Let X represent the number of heads obtained when tossing three fair coins: a red coin, a blue coin, and a green coin. Determine the value of X for each of the following outcomes:

- $X(T,T,T) = \boxed{0}$,
- $X(H,T,T) = \boxed{1}$,
- $X(T,H,T) = \boxed{1}$,
- $X(T,T,H) = \boxed{1}$,
- $X(H,H,T) = \boxed{2}$,
- $X(H,T,H) = \boxed{2}$,
- $X(T,H,H) = \boxed{2}$,
- $X(H,H,H) = \boxed{3}$.

Answer: For each outcome, X counts the number of heads (H) from the three coins:

- $X(T,T,T)$: The red coin shows tails (T), the blue coin shows tails (T), and the green coin shows tails (T). There are 0 heads, so $X(T,T,T) = 0$.
- $X(H,T,T)$: The red coin shows heads (H), the blue coin shows tails (T), and the green coin shows tails (T). There is 1 head, so $X(H,T,T) = 1$.
- $X(T,H,T)$: The red coin shows tails (T), the blue coin shows heads (H), and the green coin shows tails (T). There is 1 head, so $X(T,H,T) = 1$.
- $X(T,T,H)$: The red coin shows tails (T), the blue coin shows tails (T), and the green coin shows heads (H). There is 1 head, so $X(T,T,H) = 1$.
- $X(H,H,T)$: The red coin shows heads (H), the blue coin shows heads (H), and the green coin shows tails (T). There are 2 heads, so $X(H,H,T) = 2$.
- $X(H,T,H)$: The red coin shows heads (H), the blue coin shows tails (T), and the green coin shows heads (H). There are 2 heads, so $X(H,T,H) = 2$.
- $X(T,H,H)$: The red coin shows tails (T), the blue coin shows heads (H), and the green coin shows heads (H). There are 2 heads, so $X(T,H,H) = 2$.
- $X(H,H,H)$: The red coin shows heads (H), the blue coin shows heads (H), and the green coin shows heads (H). There are 3 heads, so $X(H,H,H) = 3$.

Ex 3: Let X represent the sum of the numbers obtained when rolling two fair six-sided dice: a red die and a blue die. Determine the value of X for each of the following outcomes:

- $X(1,1) = \boxed{2}$,
- $X(2,3) = \boxed{5}$,
- $X(3,2) = \boxed{5}$,
- $X(4,5) = \boxed{9}$,
- $X(5,4) = \boxed{9}$,
- $X(6,6) = \boxed{12}$.

Answer: For each outcome, X is the sum of the numbers on the top faces of the two dice:

- $X(1,1)$: The red die shows 1 and the blue die shows 1. The sum is $1+1=2$, so $X(1,1) = 2$.
- $X(2,3)$: The red die shows 2 and the blue die shows 3. The sum is $2+3=5$, so $X(2,3) = 5$.
- $X(3,2)$: The red die shows 3 and the blue die shows 2. The sum is $3+2=5$, so $X(3,2) = 5$.
- $X(4,5)$: The red die shows 4 and the blue die shows 5. The sum is $4+5=9$, so $X(4,5) = 9$.
- $X(5,4)$: The red die shows 5 and the blue die shows 4. The sum is $5+4=9$, so $X(5,4) = 9$.
- $X(6,6)$: The red die shows 6 and the blue die shows 6. The sum is $6+6=12$, so $X(6,6) = 12$.

Ex 4: Let X represent the product of the numbers obtained when rolling two fair six-sided dice: a red die and a blue die. Determine the value of X for each of the following outcomes:

- $X(1,1) = \boxed{1}$,
- $X(2,3) = \boxed{6}$,
- $X(3,2) = \boxed{6}$,
- $X(4,5) = \boxed{20}$,
- $X(5,4) = \boxed{20}$,
- $X(6,6) = \boxed{36}$.

Answer: For each outcome, X is the product of the numbers on the top faces of the two dice:

- $X(1,1)$: The red die shows 1 and the blue die shows 1. The product is $1 \times 1 = 1$, so $X(1,1) = 1$.
- $X(2,3)$: The red die shows 2 and the blue die shows 3. The product is $2 \times 3 = 6$, so $X(2,3) = 6$.
- $X(3,2)$: The red die shows 3 and the blue die shows 2. The product is $3 \times 2 = 6$, so $X(3,2) = 6$.
- $X(4,5)$: The red die shows 4 and the blue die shows 5. The product is $4 \times 5 = 20$, so $X(4,5) = 20$.
- $X(5,4)$: The red die shows 5 and the blue die shows 4. The product is $5 \times 4 = 20$, so $X(5,4) = 20$.
- $X(6,6)$: The red die shows 6 and the blue die shows 6. The product is $6 \times 6 = 36$, so $X(6,6) = 36$.

Ex 5: Let X represent the net gain (in dollars) from a game where you pay \$3 to play and roll one fair six-sided die. You receive a payout equal to the number of dollars shown on the die. Determine the value of X for each of the following outcomes:

- $X(1) = \boxed{-2}$,
- $X(2) = \boxed{-1}$,
- $X(3) = \boxed{0}$,
- $X(4) = \boxed{1}$,
- $X(5) = \boxed{2}$,
- $X(6) = \boxed{3}$.

Answer: For each outcome, X is the net gain, calculated as the payout (the number on the die) minus the cost to play (\$3):

- $X(1)$: The die shows 1, so the payout is \$1. The net gain is $1 - 3 = -2$, so $X(1) = -2$.
- $X(2)$: The die shows 2, so the payout is \$2. The net gain is $2 - 3 = -1$, so $X(2) = -1$.
- $X(3)$: The die shows 3, so the payout is \$3. The net gain is $3 - 3 = 0$, so $X(3) = 0$.
- $X(4)$: The die shows 4, so the payout is \$4. The net gain is $4 - 3 = 1$, so $X(4) = 1$.
- $X(5)$: The die shows 5, so the payout is \$5. The net gain is $5 - 3 = 2$, so $X(5) = 2$.
- $X(6)$: The die shows 6, so the payout is \$6. The net gain is $6 - 3 = 3$, so $X(6) = 3$.

Ex 6: Let X represent the net gain (in dollars) from a game where you pay \$2 to play and toss two fair coins: a red coin and a blue coin. You receive a payout of \$1 if no heads appear, \$3 if one head appears, and \$5 if two heads appear. Determine the value of X for each of the following outcomes:

- $X(T,T) = \boxed{-1}$,
- $X(H,T) = \boxed{1}$,
- $X(T,H) = \boxed{1}$,
- $X(H,H) = \boxed{3}$.

Answer: For each outcome, X is the net gain, calculated as the payout (based on the number of heads) minus the cost to play (\$2):

- $X(T,T)$: The red coin shows tails (T) and the blue coin shows tails (T), so there are 0 heads. The payout is \$1, and the net gain is $1 - 2 = -1$, so $X(T,T) = -1$.
- $X(H,T)$: The red coin shows heads (H) and the blue coin shows tails (T), so there is 1 head. The payout is \$3, and the net gain is $3 - 2 = 1$, so $X(H,T) = 1$.
- $X(T,H)$: The red coin shows tails (T) and the blue coin shows heads (H), so there is 1 head. The payout is \$3, and the net gain is $3 - 2 = 1$, so $X(T,H) = 1$.
- $X(H,H)$: The red coin shows heads (H) and the blue coin shows heads (H), so there are 2 heads. The payout is \$5, and the net gain is $5 - 2 = 3$, so $X(H,H) = 3$.

A.1.2 IDENTIFYING THE POSSIBLE VALUES

MCQ 7: Let X represent the number of heads obtained when tossing two fair coins: a red coin and a blue coin. Identify the possible values of X.
Choose the one correct answer:

- ☐ $\{0\}$,
- ☐ $\{0,1\}$,
- ☒ $\{0,1,2\}$,
- ☐ $\{0,1,2,3\}$.

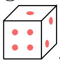
Answer: The random variable X represents the number of heads from tossing two fair coins. The possible outcomes are (T,T) (0 heads), (T,H) (1 head), (H,T) (1 head), and (H,H) (2 heads), so the possible values of X are $\{0,1,2\}$.

MCQ 8: Let X represent the number of heads obtained when tossing three fair coins: a red coin, a blue coin, and a green coin. Identify the possible values of X.
Choose the one correct answer:

- ☐ $\{0\}$,
- ☐ $\{0,1\}$,
- ☐ $\{0,1,2\}$,

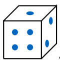
☒ $\{0, 1, 2, 3\}$.

Answer: The random variable X represents the number of heads from tossing three fair coins. The possible outcomes range from all tails (T,T,T) (0 heads) to all heads (H,H,H) (3 heads), including (T,T,H), (T,H,T), (H,T,T) (1 head), and (H,H,T), (H,T,H), (T,H,H) (2 heads). Thus, the possible values of X are $\{0, 1, 2, 3\}$.

MCQ 9: Let X represent the net gain (in dollars) from a game where you pay \$4 to play and roll one fair six-sided die . You receive a payout equal to the number of dollars shown on the die. Identify the possible values of X.
Choose the one correct answer:

- ☒ $\{-3, -2, -1, 0, 1, 2\}$,
- ☐ $\{-4\}$,
- ☐ $\{0, 1, 2, 3, 4, 5, 6\}$,
- ☐ $\{-4, -3, -2, -1, 0, 1\}$.

Answer: The random variable X represents the net gain, calculated as the payout (die outcome, 1 to 6) minus the cost (\$ 4). For a roll of 1, $X(1) = 1 - 4 = -3$; for 2, $X(2) = 2 - 4 = -2$; for 3, $X(3) = -1$; for 4, $X(4) = 0$; for 5, $X(5) = 1$; for 6, $X(6) = 2$. Thus, the possible values of X are $\{-3, -2, -1, 0, 1, 2\}$.

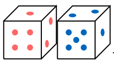
MCQ 10: Let S represent the sum of the numbers obtained when rolling two fair six-sided dice: a red die  and a blue die . Identify the possible values of S.
Choose the one correct answer:

- ☐ $\{1, 2\}$,
- ☐ $\{1, 2, 3, 4, 5, 6\}$,
- ☒ $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
- ☐ $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Answer: The random variable S represents the sum of the numbers from rolling two fair six-sided dice. The smallest sum is $1+1 = 2$ (both dice show 1), and the largest is $6+6 = 12$ (both show 6). All integer sums between 2 and 12 are possible: (1,1): 1+1=2, (1,2): 1+2=3, ..., (6,5): 6+5=11, (6,6): 6+6=12. Thus, the possible values of S are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

A.1.3 DEFINING A RANDOM VARIABLE

MCQ 11: Two fair six-sided dice (one red and one blue) are rolled, and the sum of the numbers on their top faces is recorded, as shown in this example:

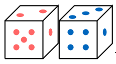
 $\longrightarrow 1 + 2 = 3$

Define the random variable S to model this situation.
Choose the one correct answer:

- ☐ The random variable S represents the difference between the numbers on the two dice: $S(i, j) = |i - j|$,
- ☐ The random variable S represents the product of the numbers on the two dice: $S(i, j) = i \times j$,
- ☒ The random variable S represents the sum of the numbers on the two dice: $S(i, j) = i + j$.

Answer: The problem states that the sum of the numbers on the top faces of the two dice is recorded, as shown by the example $1+2 = 3$. Thus, S must represent the sum, not the difference or product.

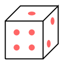
MCQ 12: Two fair six-sided dice (one red and one blue) are rolled, and the product of the numbers on their top faces is recorded, as shown in this example:

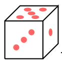
 $\longrightarrow 2 \times 3 = 6$

Define the random variable S to model this situation.
Choose the one correct answer:

- ☐ The random variable S represents the difference between the numbers on the two dice: $S(i, j) = |i - j|$,
- ☐ The random variable S represents the sum of the numbers on the two dice: $S(i, j) = i + j$,
- ☒ The random variable S represents the product of the numbers on the two dice: $S(i, j) = i \times j$.

Answer: The problem states that the product of the numbers on the top faces of the two dice is recorded, as shown by the example $2 \times 3 = 6$. Thus, S must represent the product, not the difference or sum.

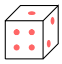
MCQ 13: A fair six-sided die  is rolled in a game where you pay \$3 to play and receive a payout equal to the number of dollars shown on the die, as shown in this example:

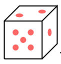
 $\longrightarrow$ Payout: \$5, Net Gain: $5 - 3 = 2$

The random variable X represents the net gain from the game.
Choose the one correct answer:

- ☐ $X(i) = 3 - i$,
- ☒ $X(i) = i - 3$,
- ☐ $X(i) = i + 3$.

Answer: The random variable X represents the net gain, which is the payout (the number on the die) minus the cost to play (\$3). The example shows a payout of \$5 with a net gain of $5 - 3 = 2$. Thus, the correct function is $X(i) = i - 3$.

MCQ 14: A fair six-sided die  is rolled in a game where you pay \$3 to play and receive a payout (in dollars) equal to 2 times the number shown on the die, as shown in this example:

 $\longrightarrow$ Payout: $2 \times 2 = 4$ dollars, Net Gain: $4 - 3 = 1$

The random variable X represents the net gain from the game.
Choose the one correct answer:

- ☒ $X(i) = 2 \times i - 3$,
- ☐ $X(i) = 3 - 2 \times i$,
- ☐ $X(i) = 2 \times i$.

Answer: The random variable X represents the net gain, which is the payout ($2 \times$ number on the die) minus the cost to play (\$3). The example shows a die roll of 2, with a payout of $2 \times 2 = 4$ dollars and a net gain of $4 - 3 = 1$. Thus, the correct function is $X(i) = 2 \times i - 3$.

A.1.4 FINDING EVENTS INVOLVING A RANDOM VARIABLE

MCQ 15: Let X be the number of heads when tossing 2 coins: and . List the outcomes for $(X = 1)$.
Choose the one correct answer:

- ☐ $\{(T, T)\}$,
- ☐ $\{(T, H)\}$,
- ☒ $\{(T, H), (H, T)\}$,
- ☐ $\{(H, H)\}$.

Answer: The random variable X represents the number of heads when tossing two coins. The possible outcomes are: (T, T) (0 heads), (T, H) (1 head), (H, T) (1 head), and (H, H) (2 heads). For $X = 1$ (one head), the outcomes are (T, H) and (H, T). Thus, the correct answer is $\{(T, H), (H, T)\}$.

MCQ 16: Let X be the number of heads when tossing 3 fair coins: a red coin , a blue coin , and a green coin . List the outcomes for $(X = 2)$.
Choose the one correct answer:

- ☐ $\{(H, H, H), (T, T, T)\}$ (all heads or all tails),
- ☒ $\{(H, H, T), (H, T, H), (T, H, H)\}$ (exactly two heads),
- ☐ $\{(T, T, H), (H, H, T)\}$ (green coin decides),
- ☐ $\{(H, H, H)\}$ (maximum heads).

Answer: The random variable X represents the number of heads when tossing three fair coins. The possible outcomes are: (T, T, T) (0 heads), (T, T, H) (1 head), (T, H, T) (1 head), (H, T, T) (1 head), (H, H, T) (2 heads), (H, T, H) (2 heads), (T, H, H) (2 heads), and (H, H, H) (3 heads). For $X = 2$ (exactly two heads), the outcomes are (H, H, T), (H, T, H), and (T, H, H). Thus, the correct answer is $\{(H, H, T), (H, T, H), (T, H, H)\}$.

MCQ 17: Let S be the sum of the numbers when rolling two fair six-sided dice: a red die and a blue die . List the outcomes for $(S = 4)$.
Choose the one correct answer:

- ☒ $\{(1, 3), (2, 2), (3, 1)\}$,
- ☐ $\{(1, 3), (2, 2)\}$,
- ☐ $\{(1, 3), (2, 2), (3, 1), (4, 0)\}$,
- ☐ $\{(4, 4)\}$.

Answer: The random variable S represents the sum of the numbers when rolling two fair six-sided dice. The possible outcomes for $S = 4$ are: $(1, 3)$ $(1 + 3 = 4)$, $(2, 2)$ $(2 + 2 = 4)$, and $(3, 1)$ $(3 + 1 = 4)$. Thus, the correct answer is $\{(1, 3), (2, 2), (3, 1)\}$.

MCQ 18: Let S be the sum of the numbers when rolling two fair six-sided dice: a red die and a blue die . List the outcomes for $(S \geq 10)$.
Choose the one correct answer:

- ☐ $\{(4, 6), (5, 5), (6, 4)\}$,
- ☐ $\{(4, 6), (5, 5)\}$,
- ☒ $\{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$,
- ☐ $\{(6, 6)\}$.

Answer: The random variable S represents the sum of the numbers when rolling two fair six-sided dice. The possible sums range from 2 to 12. For $S \geq 10$, the outcomes are: $(4, 6)$ $(4 + 6 = 10)$, $(5, 5)$ $(5 + 5 = 10)$, $(5, 6)$ $(5 + 6 = 11)$, $(6, 4)$ $(6 + 4 = 10)$, $(6, 5)$ $(6 + 5 = 11)$, and $(6, 6)$ $(6 + 6 = 12)$. Thus, the correct answer is $\{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$.

A.1.5 DEFINING A RANDOM VARIABLE TO MODEL A SITUATION

MCQ 19: In a quality control study, a manufacturing plant produces batches of 100 products each, and we count the number of defective products in each batch to assess the plant's reliability. Define the random variable X to model this situation.
Choose the one correct answer:

- ☐ The random variable X represents the total number of products in each batch, which is fixed at 100.
- ☐ The random variable X represents the plant's reliability, recorded as a binary outcome where 1 indicates a "reliable" batch (no defects) and 0 indicates an "unreliable" batch (at least one defect).
- ☒ The random variable X represents the number of defective products in a batch of 100.

Answer: The random variable X models the number of defective products in a batch of 100, ranging from 0 to 100, as this is the quantity counted. Option 1 is fixed, and option 2 is a derived binary, not the count. The correct answer is option 3.

MCQ 20: In a study of a new training program's impact, each employee's performance is assessed before and after, with improvement classified as 'Improved' or 'Not Improved'. Define the random variable Z_i, where i is the i-th employee, to model this study.
Choose the one correct answer:

- ☐ The random variable Z_i represents the number of employees in the training program.
- ☒ The random variable Z_i represents the improvement status of the i-th employee, where 1 = 'Improved' and 0 = 'Not Improved'.
- ☐ The random variable Z_i represents the duration of the training program.
- ☐ The random variable Z_i represents the performance score of the i-th employee.

Answer: The random variable Z_i models the improvement status of the i-th employee, classified as 1 for 'Improved' or 0 for 'Not Improved', based on the study's focus on this binary outcome.

MCQ 21: In a study of public transportation, a researcher records the time (in minutes) a passenger waits at a bus stop until the next bus arrives. Define the random variable T to model this situation.
Choose the one correct answer:

- ☒ The random variable T represents the waiting time (in minutes) until the next bus arrives.
- ☐ The random variable T represents the number of buses arriving per hour, which is fixed by the schedule.
- ☐ The random variable T represents whether a bus is late, recorded as 1 for "late" and 0 for "on time."

Answer: The random variable T models the waiting time (in minutes) until the next bus, as this is the measured quantity. Option 2 is a fixed rate, and option 3 is a binary outcome, not the time. The correct answer is option 1.

MCQ 22: During a soccer tournament analysis, a statistician tracks the number of goals scored by a team in a single match. Define the random variable G to model this situation.
Choose the one correct answer:

- ☐ The random variable G represents the duration of the match, fixed at 90 minutes.
- ☒ The random variable G represents the number of goals scored by the team in a match.
- ☐ The random variable G represents the match outcome, recorded as 1 for "win" and 0 for "loss or draw."

Answer: The random variable G models the number of goals scored by the team in a match, as this is the tracked quantity. Option 1 is fixed, and option 3 is a derived outcome, not the count. The correct answer is option 2.

A.2 PROBABILITY DISTRIBUTION

A.2.1 FINDING THE PROBABILITY DISTRIBUTION

Ex 23: Let X represent the number obtained when rolling one fair four-sided die with faces numbered 1, 2, 3, and 4. Complete the table with the probabilities as decimal numbers.

x	1	2	3	4
$P(X = x)$	$\boxed{0.25}$	$\boxed{0.25}$	$\boxed{0.25}$	$\boxed{0.25}$

Answer: The random variable X is the number obtained from rolling one fair four-sided die, with possible values 1, 2, 3, and 4. There are 4 outcomes with equally likely probability. The probabilities are:

- $P(X = 1) = P(\{1\})$,
  $= \frac{n(\{1\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$
- $P(X = 2) = P(\{2\})$,
  $= \frac{n(\{2\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$
- $P(X = 3) = P(\{3\})$,
  $= \frac{n(\{3\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$
- $P(X = 4) = P(\{4\})$.
  $= \frac{n(\{4\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$

The table is:

x	1	2	3	4
$P(X = x)$	0.25	0.25	0.25	0.25

Ex 24: Let X represent the number of girls in a family with two children, where each child is equally likely to be a boy or a girl. Complete the table with the probabilities as decimal numbers.

x	0	1	2
$P(X = x)$	$\boxed{0.25}$	$\boxed{0.5}$	$\boxed{0.25}$

Answer: The random variable X is the number of girls in a family with two children, with possible values 0, 1, and 2. There are $2^2 = 4$ outcomes with equally likely probability, assuming each child is a boy (B) or girl (G) with probability 1/2. The probabilities are:

- $P(X = 0) = P(\{(B, B)\})$,
  $= \frac{n(\{(B, B)\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$
- $P(X = 1) = P(\{(B, G), (G, B)\})$,
  $= \frac{n(\{(B, G), (G, B)\})}{4}$
  $= \frac{2}{4}$
  $= 0.5$
- $P(X = 2) = P(\{(G, G)\})$.
  $= \frac{n(\{(G, G)\})}{4}$
  $= \frac{1}{4}$
  $= 0.25$

The table is:

x	0	1	2
$P(X = x)$	0.25	0.5	0.25

Ex 25: Let X represent the number of heads obtained when tossing three fair coins: a red coin, a blue coin, and a green coin. Complete the table with the probabilities as decimal numbers.

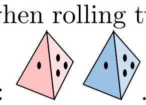
x	0	1	2	3
$P(X=x)$	$\boxed{0.125}$	$\boxed{0.375}$	$\boxed{0.375}$	$\boxed{0.125}$

Answer: The random variable X is the number of heads from tossing three fair coins, with possible values 0, 1, 2, and 3. There are $2^3 = 8$ outcomes with equally likely probability. The probabilities are:

- $P(X=0) = P(\{(T,T,T)\})$,
$$= \frac{n\left(\{(T,T,T)\}\right)}{8}$$
$$= \frac{1}{8}$$
$$= 0.125$$
- $P(X=1) = P(\{(T,T,H),(T,H,T),(H,T,T)\})$,
$$= \frac{n\left(\{(T,T,H),(T,H,T),(H,T,T)\}\right)}{8}$$
$$= \frac{3}{8}$$
$$= 0.375$$
- $P(X=2) = P(\{(H,H,T),(H,T,H),(T,H,H)\})$,
$$= \frac{n\left(\{(H,H,T),(H,T,H),(T,H,H)\}\right)}{8}$$
$$= \frac{3}{8}$$
$$= 0.375$$
- $P(X=3) = P(\{(H,H,H)\})$.
$$= \frac{n\left(\{(H,H,H)\}\right)}{8}$$
$$= \frac{1}{8}$$
$$= 0.125$$

The table is:

x	0	1	2	3
$P(X=x)$	0.125	0.375	0.375	0.125

MCQ 26: Let S represent the sum of the numbers obtained when rolling two fair four-sided dice, each numbered 1, 2, 3, and 4: . Identify the probability distribution of S.

Choose the one correct answer:

☐

s	2	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

☒

s	2	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

☐

s	2	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{5}{16}$	$\frac{6}{16}$	$\frac{7}{16}$

Answer: The random variable S is the sum of the numbers from rolling two fair four-sided dice, with possible values 2, 3, 4, 5, 6, 7, and 8. There are $4^2 = 16$ outcomes with equally likely probability. The probabilities are:

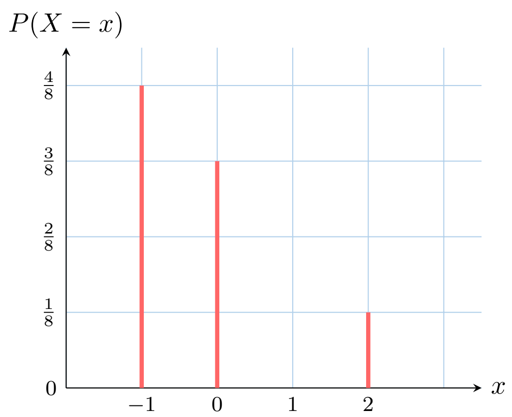
- $P(S=2) = P(\{(1,1)\})$,
$$= \frac{n\left(\{(1,1)\}\right)}{16}$$
$$= \frac{1}{16}$$
- $P(S=3) = P(\{(1,2),(2,1)\})$,
$$= \frac{n\left(\{(1,2),(2,1)\}\right)}{16}$$
$$= \frac{2}{16}$$
- $P(S=4) = P(\{(1,3),(2,2),(3,1)\})$,
$$= \frac{n\left(\{(1,3),(2,2),(3,1)\}\right)}{16}$$
$$= \frac{3}{16}$$
- $P(S=5) = P(\{(1,4),(2,3),(3,2),(4,1)\})$,
$$= \frac{n\left(\{(1,4),(2,3),(3,2),(4,1)\}\right)}{16}$$
$$= \frac{4}{16}$$
- $P(S=6) = P(\{(2,4),(3,3),(4,2)\})$,
$$= \frac{n\left(\{(2,4),(3,3),(4,2)\}\right)}{16}$$
$$= \frac{3}{16}$$
- $P(S=7) = P(\{(3,4),(4,3)\})$,
$$= \frac{n\left(\{(3,4),(4,3)\}\right)}{16}$$
$$= \frac{2}{16}$$
- $P(S=8) = P(\{(4,4)\})$.
$$= \frac{n\left(\{(4,4)\}\right)}{16}$$
$$= \frac{1}{16}$$

The table is:

s	2	3	4	5	6	7	8
$P(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

A.2.2 READING PROBABILITY DISTRIBUTIONS FROM GRAPHS

Ex 27: Consider the following probability distribution for a random variable X, represented by the graph below:

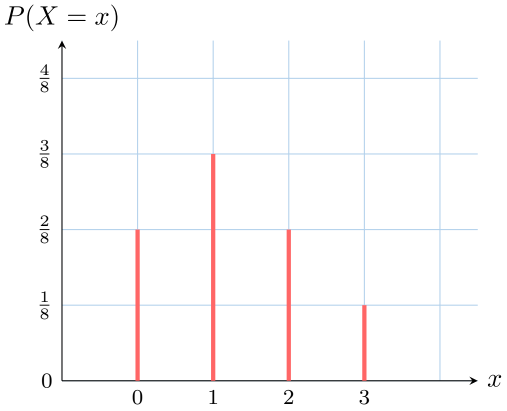


Determine the probability:

$$P(X=2) = \boxed{\frac{1}{8}}$$

Answer: The graph shows the probability distribution of X. The bar at $x = 2$ reaches a height of $\frac{1}{8}$, so $P(X = 2) = \frac{1}{8}$.

Ex 28: Consider the following probability distribution for a random variable X, represented by the graph below:

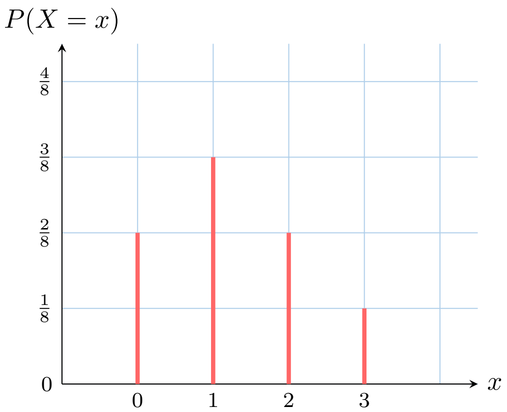


Determine the probability:

$$P(X = 1) = \boxed{\frac{3}{8}}$$

Answer: The graph shows the probability distribution of X. The bar at $x = 1$ reaches a height of $\frac{3}{8}$, so $P(X = 1) = \frac{3}{8}$.

Ex 29: Consider the following probability distribution for a random variable X, represented by the graph below:

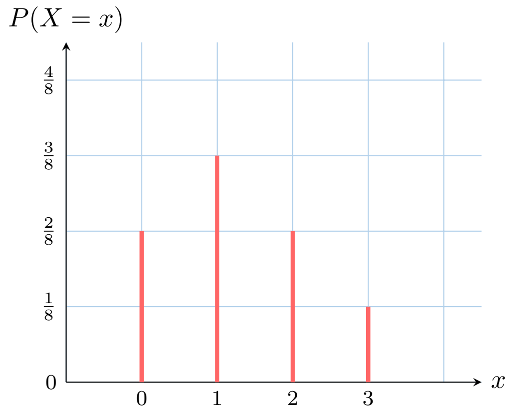


Determine the probability:

$$P(X \geq 1) = \boxed{\frac{3}{4}}$$

Answer: The graph shows the probability distribution of X. To find $P(X \geq 1)$, sum the probabilities for $x = 1, 2, 3$: $P(X = 1) = \frac{3}{8}$, $P(X = 2) = \frac{2}{8}$, $P(X = 3) = \frac{1}{8}$. Thus, $P(X \geq 1) = \frac{3}{8} + \frac{2}{8} + \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$.

Ex 30: Consider the following probability distribution for a random variable X, represented by the graph below:

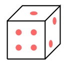
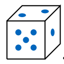


Determine the probability:

$$P(X \leq 1) = \boxed{\frac{5}{8}}$$

Answer: The graph shows the probability distribution of X. To find $P(X \leq 1)$, sum the probabilities for $x = 0$ and $x = 1$: $P(X = 0) = \frac{2}{8}$, $P(X = 1) = \frac{3}{8}$. Thus, $P(X \leq 1) = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$.

A.2.3 CONSTRUCTING A PROBABILITY DISTRIBUTION

Ex 31: Let S represent the sum of the numbers obtained when rolling two fair six-sided dice: a red die  and a blue die . Construct the probability distribution table for S.

Answer: The possible outcomes for rolling two fair six-sided dice total $6 \times 6 = 36$, each equally likely with probability $\frac{1}{36}$. The random variable S is the sum of the dice, with possible values 2 to 12 : $S(i, j) = i + j$. The values of the random variable sums are shown in this table:

die 2 / die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

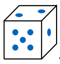
The probabilities are:

- $P(S = 2) = P(\{(1, 1)\}) = \frac{1}{36}$.
- $P(S = 3) = P(\{(1, 2), (2, 1)\}) = \frac{2}{36}$.
- $P(S = 4) = P(\{(1, 3), (2, 2), (3, 1)\}) = \frac{3}{36}$.
- $P(S = 5) = P(\{(1, 4), (2, 3), (3, 2), (4, 1)\}) = \frac{4}{36}$.
- $P(S = 6) = P(\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}) = \frac{5}{36}$.
- $P(S = 7) = P(\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}) = \frac{6}{36}$.
- $P(S = 8) = P(\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}) = \frac{5}{36}$.
- $P(S = 9) = P(\{(3, 6), (4, 5), (5, 4), (6, 3)\}) = \frac{4}{36}$.
- $P(S = 10) = P(\{(4, 6), (5, 5), (6, 4)\}) = \frac{3}{36}$.

- $P(S = 11) = P(\{(5,6),(6,5)\}) = \frac{2}{36}$.
- $P(S = 12) = P(\{(6,6)\}) = \frac{1}{36}$.

The probability distribution table is:

s	2	3	4	5	6	7	8	9	10	11	12
$P(S = s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Ex 32: Let M represent the maximum of the numbers obtained when rolling two fair six-sided dice: a red die  and a blue die . Construct the probability distribution table for M.

Answer: The possible outcomes for rolling two fair six-sided dice total $6 \times 6 = 36$, each equally likely with probability $\frac{1}{36}$. The random variable M is the maximum of the two dice: $M(i,j) = \max(i,j)$. The values of the random variable are shown in this table:

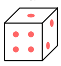
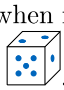
die 2 / die 1	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

The probabilities are:

- $P(M = 1) = P(\{(1,1)\}) = \frac{1}{36}$,
- $P(M = 2) = P(\{(1,2),(2,1),(2,2)\}) = \frac{3}{36}$,
- $P(M = 3) = P(\{(1,3),(2,3),(3,1),(3,2),(3,3)\}) = \frac{5}{36}$,
- $P(M = 4) = P(\{(1,4),(2,4),(3,4),(4,1),(4,2),(4,3),(4,4)\}) = \frac{7}{36}$,
- $P(M = 5) = P(\{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}) = \frac{9}{36}$,
- $P(M = 6) = P(\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}) = \frac{11}{36}$.

The probability distribution table is:

m	1	2	3	4	5	6
$P(M = m)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Ex 33: Let D represent the absolute difference of the numbers obtained when rolling two fair six-sided dice: a red die  and a blue die . Construct the probability distribution table for D.

Answer: The possible outcomes for rolling two fair six-sided dice total $6 \times 6 = 36$, each equally likely with probability $\frac{1}{36}$. The random variable D is the absolute difference of the two dice: $D(i,j) = |i - j|$. The values of the random variable are shown in this table:

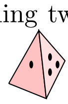
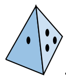
die 2 / die 1	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

The probabilities are:

- $P(D = 0) = P(\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}) = \frac{6}{36}$,
- $P(D = 1) = P(\{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,6),(6,5)\}) = \frac{10}{36}$,
- $P(D = 2) = P(\{(1,3),(2,4),(3,1),(3,5),(4,2),(4,6),(5,3),(6,4)\}) = \frac{8}{36}$,
- $P(D = 3) = P(\{(1,4),(2,5),(3,6),(4,1),(5,2),(6,3)\}) = \frac{6}{36}$,
- $P(D = 4) = P(\{(1,5),(2,6),(5,1),(6,2)\}) = \frac{4}{36}$,
- $P(D = 5) = P(\{(1,6),(6,1)\}) = \frac{2}{36}$.

The probability distribution table is:

d	0	1	2	3	4	5
$P(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

Ex 34: Let P represent the product of the numbers obtained when rolling two fair four-sided dice, each numbered 1, 2, 3, 4: a red die  and a blue die . Construct the probability distribution table for P.

Answer: The possible outcomes for rolling two fair four-sided dice total $4 \times 4 = 16$, each equally likely with probability $\frac{1}{16}$. The random variable P is the product of the two dice: $P(i,j) = i \cdot j$. The values of the random variable are shown in this table:

die 2 / die 1	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

The probabilities are:

- $P(P = 1) = P(\{(1,1)\}) = \frac{1}{16}$,
- $P(P = 2) = P(\{(1,2),(2,1)\}) = \frac{2}{16}$,
- $P(P = 3) = P(\{(1,3),(3,1)\}) = \frac{2}{16}$,
- $P(P = 4) = P(\{(1,4),(2,2),(4,1)\}) = \frac{3}{16}$,
- $P(P = 6) = P(\{(2,3),(3,2)\}) = \frac{2}{16}$,
- $P(P = 8) = P(\{(2,4),(4,2)\}) = \frac{2}{16}$,
- $P(P = 9) = P(\{(3,3)\}) = \frac{1}{16}$,
- $P(P = 12) = P(\{(3,4),(4,3)\}) = \frac{2}{16}$,
- $P(P = 16) = P(\{(4,4)\}) = \frac{1}{16}$.

The probability distribution table is:

p	1	2	3	4	6	8	9	12	16
$P(P = p)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

A.3 EXISTENCE OF A RANDOM VARIABLE WITH A GIVEN PROBABILITY DISTRIBUTION

A.3.1 VERIFYING PROBABILITY DISTRIBUTION VALIDITY

MCQ 35: Determine whether the following table represents a valid probability distribution for a random variable X:

x	1	2	3
$P(X=x)$	0.3	0.4	0.3

Select the one correct answer:

☒ Valid probability distribution

☐ Not a valid probability distribution

Answer: To check if the table represents a valid probability distribution, verify two conditions:

- For all $x = 1, 2, 3$, $P(X = x) \geq 0$. Here, $0.3 \geq 0$, $0.4 \geq 0$, and $0.3 \geq 0$, so this condition holds.
- The sum of probabilities must equal 1: $P(X = 1) + P(X = 2) + P(X = 3) = 0.3 + 0.4 + 0.3 = 1$. This condition is satisfied.
- Since both conditions are met, the probability distribution is valid.

MCQ 36: Determine whether the following table represents a valid probability distribution for a random variable X:

x	1	2	3
$P(X=x)$	0.2	0.5	0.4

Select the one correct answer:

☐ Valid probability distribution

☒ Not a valid probability distribution

Answer: To check if the table represents a valid probability distribution, verify two conditions:

- For all $x = 1, 2, 3$, $P(X = x) \geq 0$. Here, $0.2 \geq 0$, $0.5 \geq 0$, and $0.4 \geq 0$, so this condition holds.
- The sum of probabilities must equal 1: $P(X = 1) + P(X = 2) + P(X = 3) = 0.2 + 0.5 + 0.4 = 1.1$. Since $1.1 \neq 1$, this condition is not satisfied.
- Because the sum of probabilities does not equal 1, the probability distribution is not valid.

MCQ 37: Determine whether the following table represents a valid probability distribution for a random variable X:

x	1	2	3
$P(X=x)$	0.5	-0.1	0.6

Select the one correct answer:

☐ Valid probability distribution

☒ Not a valid probability distribution

Answer: To check if the table represents a valid probability distribution, verify two conditions:

- For all $x = 1, 2, 3$, $P(X = x) \geq 0$. Here, $0.5 \geq 0$, but $-0.1 < 0$, and $0.6 \geq 0$. Since $P(X = 2) = -0.1$ is negative, this condition is not satisfied.
- The sum of probabilities must equal 1: $P(X = 1) + P(X = 2) + P(X = 3) = 0.5 + (-0.1) + 0.6 = 1.0$. Although the sum equals 1, this alone does not make the distribution valid.
- Because probabilities must be non-negative and $P(X = 2) < 0$, the probability distribution is not valid.

MCQ 38: Determine whether the following table represents a valid probability distribution for a random variable X:

x	0	1	2	3
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

Select the one correct answer:

☒ Valid probability distribution

☐ Not a valid probability distribution

Answer: To check if the table represents a valid probability distribution, verify two conditions:

- For all $x = 0, 1, 2, 3$, $P(X = x) \geq 0$. Here, $\frac{1}{4} \geq 0$, $\frac{1}{8} \geq 0$, $\frac{3}{8} \geq 0$, and $\frac{1}{4} \geq 0$, so this condition holds.
- The sum of probabilities must equal 1: $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{4} + \frac{1}{8} + \frac{3}{8} + \frac{1}{4} = \frac{2}{8} + \frac{1}{8} + \frac{3}{8} + \frac{2}{8} = \frac{8}{8} = 1$. This condition is satisfied.
- Since both conditions are met, the probability distribution is valid.

A.3.2 DEFINING A RANDOM VARIABLE AND ITS PROBABILITY DISTRIBUTION

Ex 39: We survey a class of 30 students about their siblings and obtain these results: 6 students have 0 siblings, 15 have 1 sibling, 6 have 2 siblings, and 3 have 3 siblings. Let X represent the number of siblings of a student chosen at random from this class. Complete the table with the probabilities as decimal numbers.

x	0	1	2	3
$P(X=x)$	$\boxed{0.2}$	$\boxed{0.5}$	$\boxed{0.2}$	$\boxed{0.1}$

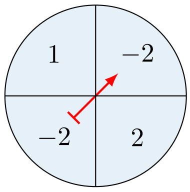
Answer: The survey includes 30 students, with X as the number of siblings: 6 students have 0 siblings, 15 have 1, 6 have 2, and 3 have 3. The probability for each value of X is the number of students with that number of siblings divided by the total (30):

- $P(X = 0) = \frac{6}{30} = \frac{1}{5} = 0.2$,
- $P(X = 1) = \frac{15}{30} = \frac{1}{2} = 0.5$,
- $P(X = 2) = \frac{6}{30} = \frac{1}{5} = 0.2$,
- $P(X = 3) = \frac{3}{30} = \frac{1}{10} = 0.1$.

Thus, the table is:

x	0	1	2	3
$P(X=x)$	0.2	0.5	0.2	0.1

Ex 40: In a game of chance, a player spins a fair spinner divided into four equal sections, each labeled with a gain (in dollars) as shown below:



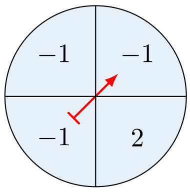
Let X represent the player's gain (in dollars) from the game. Complete the table with the probabilities as decimal numbers.

x	-2	1	2
$P(X=x)$	0.5	0.25	0.25

Answer: The spinner has four equal sections with gains -2 (twice), 1, and 2. Since it's fair, each section has a probability of $1/4 = 0.25$. The probability of $X=-2$ is $2/4 = 0.5$ (two sections), while $X=1$ and $X=2$ each have $1/4 = 0.25$ (one section each). Thus, the table is:

x	-2	1	2
$P(X=x)$	0.5	0.25	0.25

Ex 41: In a game of chance, a player spins a fair spinner divided into four equal sections, with gains (in dollars) labeled as shown below:



Let X represent the player's gain (in dollars) from the game. Complete the table with the probabilities as decimal numbers.

x	-1	2
$P(X=x)$	0.75	0.25

Answer: The spinner has four equal sections: three labeled -1 and one labeled 2. Since it's fair, each section has a probability of $1/4 = 0.25$. Thus, $P(X=-1) = 3/4 = 0.75$ (three sections), and $P(X=2) = 1/4 = 0.25$ (one section). The table is:

x	-1	2
$P(X=x)$	0.75	0.25

Ex 42: A bag contains 20 marbles: 4 red, 10 blue, 4 green, and 2 yellow. A marble is drawn at random, and points are awarded based on its color: 0 points for red, 1 point for blue, 2 points for green, and 3 points for yellow. Let X represent the number of points earned. Complete the table with the probabilities as decimal numbers.

x	0	1	2	3
$P(X=x)$	0.2	0.5	0.2	0.1

Answer: The bag has 20 marbles, with X as the points earned: 4 red marbles give 0 points, 10 blue give 1 point, 4 green give 2 points, and 2 yellow give 3 points. The probability for each value of X is the number of marbles for that point value divided by the total (20):

- $P(X=0) = \frac{4}{20} = \frac{1}{5} = 0.2$ (red),
- $P(X=1) = \frac{10}{20} = \frac{1}{2} = 0.5$ (blue),
- $P(X=2) = \frac{4}{20} = \frac{1}{5} = 0.2$ (green),
- $P(X=3) = \frac{2}{20} = \frac{1}{10} = 0.1$ (yellow).

Thus, the table is:

x	0	1	2	3
$P(X=x)$	0.2	0.5	0.2	0.1

A.3.3 FINDING PROBABILITIES IN EVERYDAY SCENARIOS

Ex 43: The random variable X represents the number of goals scored by a soccer player in a match, with the probability distribution given below:

x	0	1	2	3
$P(X=x)$	0.35	0.3	0.25	0.1

Find the probability that the player scores exactly 2 goals in a match.

The probability is $\boxed{0.25}$.

Answer:

- The probability that the player scores exactly 2 goals in a match is $P(X=2)$.
- From the distribution, $P(X=2) = 0.25$.
- Thus, the probability is 0.25.

Ex 44: The random variable X represents the number of times a student visits the nurse's office in a day, with the probability distribution given below:

x	0	1	2	3
$P(X=x)$	0.35	0.3	0.25	0.1

Find the probability that a student visits the nurse's office at least twice in a day.

The probability is $\boxed{0.35}$.

Answer:

- The probability that a student visits the nurse's office at least twice in a day is $P(X \geq 2)$.
- $$\begin{aligned} P(X \geq 2) &= P(X=2) + P(X=3) \\ &= 0.25 + 0.1 \\ &= 0.35 \end{aligned}$$
- Thus, the probability is 0.35.

Ex 45: The random variable X represents the number of rainy days in a week, with the probability distribution given below:

x	0	1	2	3
$P(X=x)$	0.35	0.3	0.25	0.1

Find the probability that there is at least one rainy day in a week.

The probability is $\boxed{0.65}$.

Answer:

- The probability that there is at least one rainy day in a week is $P(X \geq 1)$. This can be calculated in two ways:

- **Direct Method:** Sum the probabilities of all outcomes where $X \geq 1$. $P(X \geq 1) = P(X=1) + P(X=2) + P(X=3)$
$$= 0.3 + 0.25 + 0.1$$
$$= 0.65$$
- **Complement Method:** Subtract the probability of no rainy days ($X = 0$) from 1. $P(X \geq 1) = 1 - P(X=0)$
$$= 1 - 0.35$$
$$= 0.65$$
- Thus, the probability is 0.65.

Ex 46: The random variable X represents the number of defective items in a batch of 7, with the probability distribution given below:

x	0	1	2	3	4	5	6	7
$P(X=x)$	0.4	0.25	0.15	0.1	0.05	0.03	0.02	0.01

Find the probability that there is at least one defective item in the batch.

The probability is $\boxed{0.6}$.

Answer:

- The probability that there is at least one defective item in the batch is $P(X \geq 1)$. This can be calculated in two ways:
- **Direct Method:** Sum the probabilities of all outcomes where $X \geq 1$. $P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$
$$+ P(X=5) + P(X=6) + P(X=7)$$
$$= 0.25 + 0.15 + 0.1 + 0.05 + 0.03 + 0.02 + 0.01$$
$$= 0.6$$
- **Complement Method:** Subtract the probability of no defective items ($X = 0$) from 1. $P(X \geq 1) = 1 - P(X=0)$
$$= 1 - 0.4$$
$$= 0.6$$
- Thus, the probability is 0.6.

B MEASURES OF CENTER AND SPREAD

B.1 EXPECTATION

B.1.1 CALCULATING EXPECTATIONS

Ex 47: The random variable X represents the number of goals scored by a soccer player in a match, with the probability distribution given below:

x	0	1	2	3
$P(X=x)$	0.1	0.3	0.5	0.1

Calculate the expected value $E(X)$, the average number of goals scored per match.

$$E(X) = \boxed{1.6}$$

Answer:

- The expected value $E(X)$ of a probability distribution is the sum of each value multiplied by its probability.
- $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$
$$+ 3 \cdot P(X=3)$$
$$= 0 \times 0.1 + 1 \times 0.3 + 2 \times 0.5 + 3 \times 0.1$$
$$= 1.6$$
- On average, the soccer player scores 1.6 goals per match.

Ex 48: The random variable X represents the number of hours a student spends studying on a weekend, with the probability distribution given below:

x	0	1	2	3	4
$P(X=x)$	0.2	0.25	0.3	0.15	0.1

Calculate the expected value $E(X)$, the average number of hours spent studying per weekend.

$$E(X) = \boxed{1.7}$$

Answer:

- The expected value $E(X)$ of a probability distribution is the sum of each value multiplied by its probability.
- $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$
$$+ 3 \cdot P(X=3) + 4 \cdot P(X=4)$$
$$= 0 \times 0.2 + 1 \times 0.25 + 2 \times 0.3 + 3 \times 0.15 + 4 \times 0.1$$
$$= 0 + 0.25 + 0.6 + 0.45 + 0.4$$
$$= 1.7$$
- On average, the student spends 1.7 hours studying per weekend.

Ex 49: The random variable X represents the number of customers served by a cashier in an hour, with the probability distribution given below:

x	0	1	2	3	4
$P(X=x)$	0.15	0.2	0.35	0.25	0.05

Calculate the expected value $E(X)$, the average number of customers served per hour.

$$E(X) = \boxed{2}$$

Answer:

- The expected value $E(X)$ of a probability distribution is the sum of each value multiplied by its probability.
- $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2)$
$$+ 3 \cdot P(X=3) + 4 \cdot P(X=4)$$
$$= 0 \times 0.15 + 1 \times 0.2 + 2 \times 0.35 + 3 \times 0.25 + 4 \times 0.05$$
$$= 0 + 0.2 + 0.7 + 0.75 + 0.2$$
$$= 2$$
- On average, the cashier serves 2 customers per hour.

Ex 50: The random variable X represents the number of emails received by an employee in an hour, with the probability distribution given below:

x	0	1	2	3	4
$P(X=x)$	0.3	0.25	0.2	0.15	0.1

Calculate the expected value $E(X)$, the average number of emails received per hour.

$$E(X) = \boxed{1.5}$$

Answer:

- The expected value $E(X)$ of a probability distribution is the sum of each value multiplied by its probability.
- $$\begin{aligned} E(X) &= 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) \\ &\quad + 3\cdot P(X=3) + 4\cdot P(X=4) \\ &= 0\times 0.3 + 1\times 0.25 + 2\times 0.2 + 3\times 0.15 + 4\times 0.1 \\ &= 0 + 0.25 + 0.4 + 0.45 + 0.4 \\ &= 1.5 \end{aligned}$$
- On average, the employee receives 1.5 emails per hour.

B.1.2 EXPLORING EXPECTED VALUES

Ex 51: In a game of chance, a player rolls a standard six-sided die. The number rolled is the outcome of interest.

- Calculate the expected value $E(X)$ of the roll.
- Interpret the result in terms of the player's average outcome per roll.

Answer:

- Let X represent the number rolled on the die. The possible outcomes are 1, 2, 3, 4, 5, and 6, each with a probability of $\frac{1}{6}$. The probability distribution of X is:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The expected value is:

$$\begin{aligned} E(X) &= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} \\ &= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} \\ &= \frac{1+2+3+4+5+6}{6} \\ &= \frac{21}{6} \\ &= 3.5 \end{aligned}$$

- Since $E(X) = 3.5$, the player is expected to roll an average of 3.5 per roll. This value, though not a possible outcome, represents the long-term average over many rolls.

Ex 52: An insurance company offers you a contract for 150 dollars. There's a 5% chance that you will have an accident, in which case the company will pay out 2000 dollars.

- Calculate the expected profit $E(X)$ of the insurance company.
- Explain what the result means in terms of the average expected profit for the company.

Answer:

- Let X represent the insurance company's profit from your contract. The company collects a premium of 150 dollars from you. If you don't have an accident (probability 0.95 or $\frac{19}{20}$), the company pays you nothing, so its profit is 150 - 0 = 150 dollars. If you have an accident (probability 0.05 or $\frac{1}{20}$), the company pays you 2000 dollars, so its profit is 150 - 2000 = -1850 dollars. The probability distribution of X is:

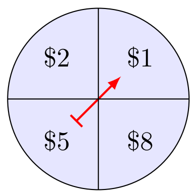
x	150	-1850
$P(X=x)$	$\frac{19}{20}$	$\frac{1}{20}$

The expected profit is:

$$\begin{aligned} E(X) &= 150\cdot\frac{19}{20} + (-1850)\cdot\frac{1}{20} \\ &= \frac{2850}{20} + \frac{-1850}{20} \\ &= \frac{2850-1850}{20} \\ &= \frac{1000}{20} \\ &= 50 \end{aligned}$$

- Since $E(X) = 50$, the insurance company expects to make an average profit of 50 dollars for each contract like yours. This positive expected profit means the company can cover its expenses, such as salaries and administrative costs, and still have money left over as profit.

Ex 53: In a game of chance, a player spins a spinner:



The player wins the amount of money indicated by the arrow, but it costs \$5 to play each game. In gambling, the gain is defined as the payout minus the cost to play.

- Calculate the expected gain $E(X)$ of the player.
- Interpret the result in terms of the player's average outcome per game.

Answer:

- Let X represent the player's gain per game. The possible gains are -4 (\$1 payout), -3 (\$2 payout), 0 (\$5 payout), and 3 (\$8 payout), after subtracting the \$5 cost to play. Each outcome has a probability of $\frac{1}{4}$. The probability distribution of X is:

x	-4	-3	0	3
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

The expected gain is:

$$\begin{aligned} E(X) &= (-4)\cdot\frac{1}{4} + (-3)\cdot\frac{1}{4} + 0\cdot\frac{1}{4} + 3\cdot\frac{1}{4} \\ &= \frac{-4}{4} + \frac{-3}{4} + \frac{0}{4} + \frac{3}{4} \\ &= -1 - 0.75 + 0 + 0.75 \\ &= -1 \end{aligned}$$

- Since $E(X) = -1$, the player is expected to lose \$1 on average per game.

Ex 54: In a game of chance, a player bets \$1 on a single number in a classical roulette wheel numbered from 0 to 36. If the chosen number comes up, the player wins 35 times their bet plus their bet back, receiving a total payout of \$36; otherwise, they lose their bet. In gambling, the gain is defined as the payout minus the cost to play.

- Calculate the expected gain $E(X)$ of the player.
- Interpret the result in terms of the player's average outcome per game.

Answer:

- Let X represent the player's gain per game. The player bets \$1 on one of the 37 numbers (0 to 36). If they win (probability $\frac{1}{37}$), they receive \$36 (35 times their bet plus the bet back), so the gain is $\$36 - \$1 = 35$. If they lose (probability $\frac{36}{37}$), they receive nothing, so the gain is $\$0 - \$1 = -1$. The probability distribution of X is:

x	-1	35
$P(X=x)$	$\frac{36}{37}$	$\frac{1}{37}$

The expected gain is: $$\begin{aligned} E(X) &= (-1)\cdot\frac{36}{37} + 35\cdot\frac{1}{37} \\ &= \frac{-36}{37} + \frac{35}{37} \\ &= \frac{-36+35}{37} \\ &= -\frac{1}{37} \\ &\approx -0.0270 \end{aligned}$$

- Since $E(X) = -\frac{1}{37} \approx -0.0270$, the player is expected to lose approximately \$0.027 (or about 2.7 cents) on average per game. This negative expected value reflects the house edge in roulette.

B.2 VARIANCE AND STANDARD DEVIATION

B.2.1 CALCULATING STANDARD DEVIATION

Ex 55: A soccer player's number of goals scored in a match is represented by the random variable X. The probability distribution for X is shown below:

x (goals)	0	1	2	3
$P(X=x)$	0.6	0.1	0.1	0.2

Calculate the standard deviation $\sigma(X)$, which shows how much the number of goals typically varies from the average per match (round at two decimal places).

$$\sigma(X) = \boxed{1.22}$$

Answer:

- Calculate the expectation (mean):

$$\begin{aligned} E(X) &= \sum x\cdot P(X=x) \\ &= (0\cdot 0.6) + (1\cdot 0.1) + (2\cdot 0.1) + (3\cdot 0.2) \\ &= 0 + 0.1 + 0.2 + 0.6 \\ &= 0.9 \end{aligned}$$

- Calculate the variance using the computational formula $V(X) = E(X^2) - [E(X)]^2$:

$$\begin{aligned} E(X^2) &= \sum x^2\cdot P(X=x) \\ &= (0^2\cdot 0.6) + (1^2\cdot 0.1) + (2^2\cdot 0.1) + (3^2\cdot 0.2) \\ &= 0 + 0.1 + 0.4 + 1.8 \\ &= 2.3 \end{aligned}$$

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= 2.3 - (0.9)^2 \\ &= 2.3 - 0.81 \\ &= 1.49 \end{aligned}$$

- Calculate the standard deviation:

$$\begin{aligned} \sigma(X) &= \sqrt{V(X)} \\ &= \sqrt{1.49} \\ &\approx 1.2206... \end{aligned}$$

Rounding to two decimal places, $\sigma(X) \approx 1.22$.

Ex 56: The random variable X represents the number of cups of coffee a teacher drinks in a day. The probability distribution for X is shown below:

x (cups)	0	1	2	3
$P(X=x)$	0.1	0.3	0.4	0.2

Calculate the standard deviation $\sigma(X)$, which shows how much the number of cups typically varies from the average per day (round to two decimal places).

$$\sigma(X) = \boxed{0.90}$$

Answer:

- Calculate the expectation:

$$\begin{aligned} E(X) &= \sum xP(X=x) \\ &= 0\cdot 0.1 + 1\cdot 0.3 + 2\cdot 0.4 + 3\cdot 0.2 \\ &= 1.7 \end{aligned}$$

- Calculate the variance:

$$\begin{aligned} V(X) &= \sum(x - E(X))^2 P(X=x) \\ &= (0-1.7)^2\cdot 0.1 + (1-1.7)^2\cdot 0.3 + (2-1.7)^2\cdot 0.4 \\ &\quad + (3-1.7)^2\cdot 0.2 \\ &= 0.289 + 0.147 + 0.036 + 0.338 \\ &= 0.81 \end{aligned}$$

- Calculate the standard deviation:

$$\begin{aligned}\sigma(X) &= \sqrt{V(X)}\\ &= \sqrt{0.81}\\ &= 0.90\end{aligned}$$

(Using a calculator, $\sqrt{0.81} = 0.9$, exact.)

- The standard deviation $\sigma(X) = 0.90$ tells us that the number of cups of coffee typically varies by about 0.9 from the average of 1.7 cups per day. This suggests the teacher's coffee consumption is fairly consistent, usually staying close to the average.

Ex 57: The random variable X represents the number of siblings a student in a class has. The probability distribution for X is shown below:

x (siblings)	0	1	2	3
$P(X = x)$	0.3	0.4	0.2	0.1

Calculate the standard deviation $\sigma(X)$, which shows how much the number of siblings typically varies from the average per student (round to two decimal places).

$$\sigma(X) = \boxed{0.94}$$

Answer:

- Calculate the expectation:

$$\begin{aligned}E(X) &= \sum xP(X = x)\\ &= 0 \cdot 0.3 + 1 \cdot 0.4 + 2 \cdot 0.2 + 3 \cdot 0.1\\ &= 0 + 0.4 + 0.4 + 0.3\\ &= 1.1\end{aligned}$$

- Calculate the variance:

$$\begin{aligned}V(X) &= \sum (x - E(X))^2 P(X = x)\\ &= (0 - 1.1)^2 \cdot 0.3 + (1 - 1.1)^2 \cdot 0.4 + (2 - 1.1)^2 \cdot 0.2\\ &\quad + (3 - 1.1)^2 \cdot 0.1\\ &= 0.89\end{aligned}$$

- Calculate the standard deviation:

$$\begin{aligned}\sigma(X) &= \sqrt{V(X)}\\ &= \sqrt{0.89}\\ &\approx 0.94\end{aligned}$$

(Using a calculator, $\sqrt{0.89} \approx 0.9433$, rounded to 0.94.)

- The standard deviation $\sigma(X) \approx 0.94$ tells us that the number of siblings typically varies by about 0.94 from the average of 1.1 siblings per student. This suggests most students have a number of siblings close to the average, with moderate variation.

Ex 58: The random variable X represents the number of car accidents a driver has in a year. The probability distribution for X is shown below:

x (accidents)	0	1	2	3
$P(X = x)$	0.7	0.2	0.08	0.02

Calculate the standard deviation $\sigma(X)$, which shows how much the number of car accidents typically varies from the average per year (round to two decimal places).

$$\sigma(X) = \boxed{0.72}$$

Answer:

- Calculate the expectation:

$$\begin{aligned}E(X) &= \sum xP(X = x)\\ &= 0 \cdot 0.7 + 1 \cdot 0.2 + 2 \cdot 0.08 + 3 \cdot 0.02\\ &= 0 + 0.2 + 0.16 + 0.06\\ &= 0.42\end{aligned}$$

- Calculate the variance:

$$\begin{aligned}V(X) &= \sum (x - E(X))^2 P(X = x)\\ &= (0 - 0.42)^2 \cdot 0.7 + (1 - 0.42)^2 \cdot 0.2 + (2 - 0.42)^2 \cdot 0.08\\ &\quad + (3 - 0.42)^2 \cdot 0.02\\ &= 0.523608\end{aligned}$$

- Calculate the standard deviation:

$$\begin{aligned}\sigma(X) &= \sqrt{V(X)}\\ &= \sqrt{0.523608}\\ &\approx 0.72\end{aligned}$$

(Using a calculator, $\sqrt{0.523608} \approx 0.7236$, rounded to 0.72.)

- The standard deviation $\sigma(X) \approx 0.72$ tells us that the number of car accidents typically varies by about 0.72 from the average of 0.42 accidents per year. This suggests that most drivers have very few accidents (often 0), with occasional higher numbers causing some variation.

B.2.2 CHOOSING BASED ON EXPECTED VALUES AND RISK

Ex 59: A customer is choosing between two financial options for a \$1000 investment over one year. The random variables X and Y represent the gain (in dollars) for Option A and Option B, respectively, with:

- Expected gain: $E(X) = 67.5$ for Option A and $E(Y) = 120$ for Option B
- Standard deviation: $\sigma(X) = 16.01$ for Option A and $\sigma(Y) = 95.39$ for Option B

The customer wants to minimize risk (safe placement). Which option should they choose, and why? Justify your answer based on the given expected gains and standard deviations.

Answer: The customer should choose **Option A**, the safe option. Here's why:

- **Expected Gain Comparison**: Option A has an expected gain of \$67.5, while Option B has a higher expected gain of \$120. This means Option B offers a greater average return on the \$1000 investment.

- **Standard Deviation Comparison**: Option A has a standard deviation of 16.01, which is much lower than Option B's standard deviation of 95.39. The standard deviation measures the risk or variability in the gains. A lower standard deviation means the gains are more consistent and closer to the expected value, while a higher standard deviation indicates more uncertainty, including the possibility of losses.
- **Customer's Priority**: The customer wants to minimize risk, which is defined as choosing the option with the lowest standard deviation. Option A's $\sigma(X) = 16.01$ is significantly less than Option B's $\sigma(Y) = 95.39$, making it the less risky choice.

Thus, despite Option B's higher expected gain, the customer chooses **Option A** because its lower standard deviation (16.01 vs. 95.39) indicates lower risk, which is their primary concern.

Ex 60: A customer is choosing between two financial options for a \$1000 investment over one year. The random variables X and Y represent the gain (in dollars) for Option A and Option B, respectively, with:

- Expected gain: $E(X) = 67.5$ for Option A and $E(Y) = 120$ for Option B
- Standard deviation: $\sigma(X) = 16.01$ for Option A and $\sigma(Y) = 95.39$ for Option B

The customer wants to maximize profit regardless of the risk. Which option should they choose, and why? Justify your answer based on the given expected gains and standard deviations.

Answer: The customer should choose **Option B**, the risky option. Here's why:

- **Expected Gain Comparison**: Option A has an expected gain of \$67.5, while Option B has a higher expected gain of \$120. This means Option B offers a greater average return on the \$1000 investment, making it more profitable on average.
- **Standard Deviation Comparison**: Option A has a standard deviation of 16.01, which is much lower than Option B's standard deviation of 95.39. The standard deviation measures the risk or variability in the gains. A lower standard deviation indicates more consistent gains, while a higher standard deviation suggests greater uncertainty, including potential for both higher gains and losses.
- **Customer's Priority**: The customer wants to maximize profit regardless of the risk, which means choosing the option with the highest expected gain. Option B's $E(Y) = 120$ is significantly greater than Option A's $E(X) = 67.5$, aligning with their goal of maximizing profit over minimizing risk.

Thus, because Option B has a higher expected gain (\$120 vs. \$67.5), the customer chooses **Option B** despite its higher standard deviation (95.39 vs. 16.01), as maximizing profit is their primary concern, not risk.

Ex 61: A person is choosing between two countries to live in based on the average annual temperature (in degrees Celsius). The random variables X and Y represent the temperature for Country A and Country B, respectively, with:

- Expected temperature: $E(X) = 18.5$ for Country A and $E(Y) = 25.0$ for Country B
- Standard deviation: $\sigma(X) = 3.2$ for Country A and $\sigma(Y) = 8.5$ for Country B

The person wants to maximize warmth regardless of temperature variability. Which country should they choose, and why? Justify your answer based on the given expected temperatures and standard deviations.

Answer: The person should choose **Country B**, the warmer option. Here's why:

- **Expected Temperature Comparison**: Country A has an expected temperature of 18.5°C, while Country B has a higher expected temperature of 25.0°C. This means Country B is warmer on average, offering more heat throughout the year.
- **Standard Deviation Comparison**: Country A has a standard deviation of 3.2, which is much lower than Country B's standard deviation of 8.5. The standard deviation measures the variability in temperature. A lower standard deviation means the temperature is more consistent, while a higher standard deviation indicates larger fluctuations, possibly including colder days.
- **Person's Priority**: The person wants to maximize warmth regardless of temperature variability, which means choosing the country with the highest expected temperature. Country B's $E(Y) = 25.0$ is significantly greater than Country A's $E(X) = 18.5$, aligning with their goal of maximizing warmth over minimizing variability.

Thus, because Country B has a higher expected temperature (25.0°C vs. 18.5°C), the person chooses **Country B** despite its higher standard deviation (8.5 vs. 3.2), as maximizing warmth is their primary concern, not variability.

Ex 62: A student is choosing between two part-time jobs based on weekly earnings (in dollars). The random variables X and Y represent the earnings for Job A and Job B, respectively, with:

- Expected earnings: $E(X) = 150$ for Job A and $E(Y) = 175$ for Job B
- Standard deviation: $\sigma(X) = 10.5$ for Job A and $\sigma(Y) = 25.8$ for Job B

The student wants to minimize variability in their earnings (stable income). Which job should they choose, and why? Justify your answer based on the given expected earnings and standard deviations.

Answer: The student should choose **Job A**, the stable option. Here's why:

- **Expected Earnings Comparison**: Job A has an expected earnings of \$150, while Job B has a higher expected earnings of \$175. This means Job B offers a greater average income per week.
- **Standard Deviation Comparison**: Job A has a standard deviation of 10.5, which is much lower than Job B's standard deviation of 25.8. The standard deviation measures the variability in earnings. A lower standard deviation means the weekly earnings are more consistent and predictable, while a higher standard deviation indicates larger fluctuations, which could mean less reliability.

- **Student's Priority**: The student wants to minimize variability in their earnings, which is defined as choosing the job with the lowest standard deviation. Job A's $\sigma(X) = 10.5$ is significantly less than Job B's $\sigma(Y) = 25.8$, making it the more stable choice.

Thus, despite Job B's higher expected earnings (\$175 vs. \$150), the student chooses **Job A** because its lower standard deviation (10.5 vs. 25.8) indicates less variability, which is their primary concern.

C CLASSICAL DISTRIBUTIONS

C.1 UNIFORM DISTRIBUTION

C.1.1 IDENTIFYING UNIFORM DISTRIBUTIONS

MCQ 63: When rolling a fair six-sided die, the random variable X represents the number that appears on the top face.
Does the random variable X follow a uniform distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- For a fair six-sided die, each outcome (1, 2, 3, 4, 5, 6) has an equal probability of $\frac{1}{6}$.
- A uniform distribution occurs when all possible values of a random variable have the same probability.
- Thus, X follows a uniform distribution, so the correct answer is **True**.

MCQ 64: A standard deck of 52 playing cards is divided into 4 suits: hearts, diamonds, clubs, and spades, with each suit containing 13 cards. A card is drawn at random from the deck, and the random variable Z represents the suit of the card, assigned as follows: 1 for hearts, 2 for diamonds, 3 for clubs, and 4 for spades.
Does the random variable Z follow a uniform distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- In a standard deck of 52 cards, each of the 4 suits (hearts, diamonds, clubs, spades) has 13 cards, so the probability of drawing any one suit is $\frac{13}{52} = \frac{1}{4}$.
- A uniform distribution occurs when all possible values of a random variable have equal probabilities.
- Since each value of Z (1, 2, 3, 4) has the same probability ($\frac{1}{4}$), Z follows a uniform distribution. Thus, the correct answer is **True**.

MCQ 65: A bag contains 5 red, 4 blue, and 3 green marbles, making a total of 12 marbles. A marble is drawn at random, and the random variable Y represents the color of the marble, assigned as follows: 1 for red, 2 for blue, and 3 for green.
Does the random variable Y follow a uniform distribution?
Select the one correct answer:

☐ True

☒ False

Answer:

- The bag has 12 marbles: 5 red, 4 blue, and 3 green. The probabilities are $\frac{5}{12}$ for red, $\frac{4}{12} = \frac{1}{3}$ for blue, and $\frac{3}{12} = \frac{1}{4}$ for green, which are not equal.
- A uniform distribution requires all possible values of a random variable to have the same probability.
- Since the probabilities of Y (1, 2, 3) differ ($\frac{5}{12} \neq \frac{1}{3} \neq \frac{1}{4}$), Y does not follow a uniform distribution. Thus, the correct answer is **False**.

MCQ 66: A telephone number is selected at random, and the random variable X represents its last digit, which can be any integer from 0 to 9.
Does the random variable X follow a uniform distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- When a telephone number is selected at random, the last digit can be any value from 0 to 9, and there are 10 possible outcomes. Assuming no restrictions or biases (e.g., no exclusion of certain digits), each digit has an equal probability of $\frac{1}{10}$.
- A uniform distribution occurs when all possible values of a random variable have the same probability.
- Since each value of X (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) has the same probability ($\frac{1}{10}$), X follows a uniform distribution. Thus, the correct answer is **True**.

C.1.2 EXPECTED VALUE AND VARIANCE FOR UNIFORM DISTRIBUTIONS

Ex 67: A fair eight-sided die, with faces numbered consecutively from 1 to 8, is rolled. Let X be the number shown on the top face.

1. State the distribution of X.
2. Find the expected value, $E(X)$.
3. Find the variance, $V(X)$.

Answer:

1. Since the die is fair, each of the $n = 8$ outcomes is equally likely. Therefore, X follows a discrete uniform distribution on the set $\{1, 2, \ldots, 8\}$.
2. The expected value is calculated using the formula $E(X) = \frac{n+1}{2}$:
$$E(X) = \frac{8+1}{2} = \frac{9}{2} = 4.5$$

3. The variance is calculated using the formula $V(X) = \frac{n^2-1}{12}$:

$$V(X) = \frac{8^2 - 1}{12} = \frac{64-1}{12} = \frac{63}{12} = 5.25$$

Ex 68: The random variable Y follows a discrete uniform distribution on the set of integers $\{1, 2, \ldots, n\}$. The expected value of Y is $E(Y) = 10$.

1. Find the value of n.
2. Hence, find the variance of Y.

Answer:

1. We use the formula for the expected value and solve for n:

$$\begin{aligned} E(Y) &= \frac{n+1}{2} \\ 10 &= \frac{n+1}{2} \\ 20 &= n+1 \\ n &= 19 \end{aligned}$$

2. Now that we know $n = 19$, we can find the variance:

$$\begin{aligned} V(Y) &= \frac{n^2-1}{12} \\ &= \frac{19^2-1}{12} \\ &= \frac{361-1}{12} \\ &= \frac{360}{12} = 30 \end{aligned}$$

Ex 69: A spinner is designed to land on any integer from 5 to 14 inclusive, with each outcome being equally likely. Let X be the number the spinner lands on. Find $E(X)$ and $V(X)$.

Answer: The standard formulas are for a distribution on $\{1, \ldots, n\}$. The set of outcomes for X is $\{5, 6, \ldots, 14\}$.
Let's define a new random variable $Y = X - 4$. The possible values for Y are $\{1, 2, \ldots, 10\}$.Y follows a discrete uniform distribution with $n = 10$.
We can find the mean and variance of Y:

$$E(Y) = \frac{10+1}{2} = 5.5$$

$$V(Y) = \frac{10^2-1}{12} = \frac{99}{12} = 8.25$$

Now we use the properties of expectation and variance to find them for X. Since $X = Y + 4$:

$$E(X) = E(Y+4) = E(Y) + 4 = 5.5 + 4 = 9.5$$

$$V(X) = V(Y+4) = V(Y) = 8.25$$

Ex 70: Two games are proposed at a fair. In both games, the number of points you win is the number shown on the die you roll.

- Game A: Roll a fair 6-sided die (X_A).
- Game B: Roll a fair 4-sided die (X_B).

Which game offers a higher average score? Which game is more predictable (less variable)? Justify your answers with calculations.

Answer: We calculate the expected value and variance for each game. Both are discrete uniform distributions.

- **Game A ($n = 6$):**

$$E(X_A) = \frac{6+1}{2} = 3.5$$

$$V(X_A) = \frac{6^2-1}{12} = \frac{35}{12} \approx 2.92$$

- **Game B ($n = 4$):**

$$E(X_B) = \frac{4+1}{2} = 2.5$$

$$V(X_B) = \frac{4^2-1}{12} = \frac{15}{12} = 1.25$$

Conclusion:

- **Average Score:** Since $E(X_A) = 3.5$ is greater than $E(X_B) = 2.5$, **Game A offers a higher average score**.
- **Predictability:** Predictability is measured by variance (a smaller variance means less spread and more predictable outcomes). Since $V(X_B) = 1.25$ is less than $V(X_A) \approx 2.92$, **Game B is more predictable**.

C.2 BERNOULLI DISTRIBUTION

C.2.1 IDENTIFYING BERNOULLI DISTRIBUTIONS

MCQ 71: A fair coin is tossed once, and the random variable X represents the outcome, with values assigned as follows: 0 for tails and 1 for heads.
Does the random variable X follow a Bernoulli distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- When a fair coin is tossed, the random variable X has exactly two possible values: 0 (tails) or 1 (heads), each with a probability of $\frac{1}{2}$.
- A Bernoulli distribution describes a random variable with two possible values, typically labeled 0 and 1, where the probability of success (1) is p and failure (0) is $1-p$. Here, $p = \frac{1}{2}$.
- Since X fits this definition with two outcomes and equal probabilities, it follows a Bernoulli distribution. Thus, the correct answer is **True**.

MCQ 72: Two fair coins are tossed, and the random variable X represents the number of heads obtained, with possible values 0, 1, or 2.
Does the random variable X follow a Bernoulli distribution?
Select the one correct answer:

☐ True

☒ False

Answer:

- When two fair coins are tossed, the random variable X (number of heads) can take three possible values: 0 (no heads), 1 (one head), or 2 (two heads), with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively.
- A Bernoulli distribution describes a random variable with exactly two possible outcomes (typically 0 and 1), representing a single trial with a success probability p and failure probability $1 - p$.
- Since X has three possible outcomes (0, 1, 2) from two trials, it does not fit the Bernoulli distribution, which is limited to one trial and two outcomes. Instead, X follows a binomial distribution with $n = 2$ and $p = \frac{1}{2}$. Thus, the correct answer is **False**.

MCQ 73: Mia asks a boy, "Do you like me?" and he answers with either "Yes" or "No." The random variable X represents his response, with values assigned as follows: 1 for "Yes" and 0 for "No." Assume the boy's answer is a random guess with equal chances for each response.
Does the random variable X follow a Bernoulli distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- When Mia asks the boy, the random variable X has exactly two possible outcomes: 1 ("Yes") or 0 ("No"). Since his answer is a random guess with equal chances, each outcome has a probability of $\frac{1}{2}$.
- A Bernoulli distribution describes a random variable with two possible outcomes (typically 0 and 1) from a single trial, where the probability of success (1) is p and failure (0) is $1 - p$. Here, $p = \frac{1}{2}$.
- Since X matches this definition—two outcomes from one question with $p = \frac{1}{2}$—it follows a Bernoulli distribution. Thus, the correct answer is **True**.

MCQ 74: A student is selected at random from a class containing 5 girls and 10 boys, making a total of 15 students. The random variable X represents the gender of the selected student, with values assigned as follows: 0 for boy and 1 for girl.
Does the random variable X follow a Bernoulli distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- When a student is selected at random from the class, X has two possible outcomes: 0 (boy) with probability $\frac{10}{15} = \frac{2}{3}$ or 1 (girl) with probability $\frac{5}{15} = \frac{1}{3}$.
- A Bernoulli distribution describes a random variable with exactly two outcomes (typically 0 and 1) from a single trial, where the probability of success (1) is p and failure (0) is $1 - p$. Here, $p = \frac{1}{3}$.
- Since X has two outcomes from one selection and fits the Bernoulli definition (with $p = \frac{1}{3}$), it follows a Bernoulli distribution. Thus, the correct answer is **True**.

C.2.2 FINDING A PROBABILITY OF SUCCESS

Ex 75: A fair coin is tossed once, and the random variable X represents the outcome, with values assigned as follows: 0 for tails and 1 for heads.
Find the probability of success for the random variable X.

$$p = \boxed{\frac{1}{2}}$$

Answer:

- The random variable X represents the outcome of a fair coin toss, where $X = 0$ for tails and $X = 1$ for heads. In a Bernoulli distribution, "success" is typically defined as $X = 1$ (heads).
- For a fair coin, there are 2 equally likely outcomes: tails or heads. The probability of getting heads (success, $X = 1$) is $\frac{1}{2}$.
- The probability of getting tails (failure, $X = 0$) is also $\frac{1}{2}$, but this is not the success probability.
- Thus, the probability of success for X is $p = \frac{1}{2}$.

Ex 76: A student is selected at random from a class containing 5 girls and 10 boys, making a total of 15 students. The random variable X represents the gender of the selected student, with values assigned as follows: 0 for boy and 1 for girl.
Find the probability of success for the random variable X.

$$p = \boxed{\frac{1}{3}}$$

Answer:

- The random variable X represents the gender of a randomly selected student, where $X = 0$ for a boy and $X = 1$ for a girl. In a Bernoulli distribution, "success" is typically defined as $X = 1$.
- The class has 15 students: 5 girls and 10 boys. The probability of selecting a girl (success, $X = 1$) is the number of girls divided by the total number of students: $\frac{5}{15} = \frac{1}{3}$.
- The probability of selecting a boy (failure, $X = 0$) is $\frac{10}{15} = \frac{2}{3}$, but this is not the success probability.
- Thus, the probability of success for X is $p = \frac{1}{3}$.

Ex 77: A basketball player has made 300 successful shots out of 400 attempts in practice. For a single shot, the random variable X represents the outcome, with values assigned as follows: 0 for a miss and 1 for a success.
Find the probability of success for the random variable X.

$$p = \boxed{\frac{3}{4}}$$

Answer:

- The random variable X represents the outcome of a single shot by the basketball player, where $X = 0$ for a miss and $X = 1$ for a success. In a Bernoulli distribution, "success" is typically defined as $X = 1$.
- Based on past performance, the player has made 300 successful shots out of 400 attempts. The probability of success ($X = 1$) is the number of successful shots divided by the total attempts: $\frac{300}{400} = \frac{3}{4}$.
- The probability of a miss (failure, $X = 0$) is $\frac{100}{400} = \frac{1}{4}$, but this is not the success probability.
- Thus, the probability of success for X is $p = \frac{3}{4}$.

Ex 78: A student, Leo, guesses the answer to a true/false quiz question. He has a tendency to guess "True" 60% of the time and "False" 40% of the time. The random variable X represents the outcome of his guess, with values assigned as follows: 0 for "False" and 1 for "True."
Find the probability of success for the random variable X.

$$p = \boxed{\frac{3}{5}}$$

Answer:

- The random variable X represents the outcome of Leo's guess on a true/false question, where $X = 0$ for "False" and $X = 1$ for "True." In a Bernoulli distribution, "success" is typically defined as $X = 1$ ("True").
- Leo guesses "True" 60% of the time and "False" 40% of the time. The probability of guessing "True" (success, $X = 1$) is given as 60%, or $\frac{60}{100} = \frac{3}{5}$.
- The probability of guessing "False" (failure, $X = 0$) is 40%, or $\frac{40}{100} = \frac{2}{5}$, but this is not the success probability.
- Thus, the probability of success for X is $p = \frac{3}{5}$.

C.3 BINOMIAL DISTRIBUTION

C.3.1 IDENTIFYING BINOMIAL DISTRIBUTIONS

MCQ 79: A fair coin is tossed 100 times independently, and the random variable X represents the number of heads obtained, with possible values ranging from 0 to 100.
Does the random variable X follow a binomial distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- Tossing a fair coin once is a Bernoulli trial with two outcomes: heads (success) or tails (failure), each with probability $\frac{1}{2}$. Here, the experiment is repeated 100 times independently.
- The random variable X counts the number of successes (heads) across these 100 trials, taking integer values from 0 (no heads) to 100 (all heads).
- A binomial distribution models the number of successes in n independent Bernoulli trials with a constant success probability p. For X, $n = 100$ and $p = \frac{1}{2}$, so X follows a binomial distribution. Thus, the correct answer is **True**.

MCQ 80: An urn contains 3 red balls and 7 blue balls. A ball is drawn at random and replaced 50 times independently, and the random variable X represents the number of red balls drawn, with possible values ranging from 0 to 50.
Does the random variable X follow a binomial distribution?
Select the one correct answer:

☒ True

☐ False

Answer:

- Drawing a ball from the urn with replacement is a Bernoulli trial with two outcomes: red (success) or blue (failure). The probability of drawing a red ball is $\frac{3}{10}$ (3 red out of 10 total), and this remains constant due to replacement. Here, the experiment is repeated 50 times independently.
- The random variable X counts the number of successes (red balls) across these 50 trials, taking integer values from 0 (no red balls) to 50 (all red balls).
- A binomial distribution models the number of successes in n independent Bernoulli trials with a constant success probability p. For X, $n = 50$ and $p = \frac{3}{10}$, so X follows a binomial distribution. Thus, the correct answer is **True**.

MCQ 81: An urn contains 3 red balls and 7 blue balls, making a total of 10 balls. A ball is drawn at random without replacement 5 times, and the random variable X represents the number of red balls drawn, with possible values ranging from 0 to 3.
Does the random variable X follow a binomial distribution?
Select the one correct answer:

☐ True

☒ False

Answer:

- Drawing a ball from the urn without replacement changes the probabilities for each subsequent draw. Initially, the probability of drawing a red ball is $\frac{3}{10}$, but after drawing one (e.g., a red), it becomes $\frac{2}{9}$ or $\frac{3}{9}$ depending on the outcome. Thus, the trials are not independent, and the success probability is not constant.
- The random variable X counts the number of red balls (successes) over 5 draws, with values from 0 to 3 (since only 3 red balls exist).
- A binomial distribution requires n independent Bernoulli trials with a constant success probability p. Here, the lack of replacement violates independence and constancy of p, so X does not follow a binomial distribution (it follows a hypergeometric distribution instead). Thus, the correct answer is **False**.

MCQ 82: In a group of 10 friends, a student invites 5 friends one by one to a party. Initially, each friend has a 40% chance of accepting, but for each friend who says "Yes," the next friend's probability of accepting increases by 5% due to excitement about the party. The random variable X represents the number of friends who accept, with possible values ranging from 0 to 5. Does the random variable X follow a binomial distribution?
Select the one correct answer:

☐ True

☒ False

Answer:

- When the student invites the first friend, the probability of a "Yes" is 40% ($\frac{2}{5}$). If they accept, the next friend's chance rises to 45% ($\frac{9}{20}$) due to excitement, then to 50% ($\frac{1}{2}$) with another "Yes," and so on. Each "Yes" changes the success probability for the next invitation, so it's not constant across the 5 trials.
- The random variable X counts how many of the 5 invited friends accept, with values from 0 (all decline) to 5 (all accept).
- A binomial distribution requires n independent Bernoulli trials with a constant success probability p. Here, the probability of acceptance increases with each "Yes," violating the constant p condition (and potentially introducing dependence), so X does not follow a binomial distribution. Thus, the correct answer is **False**.

C.3.2 FINDING PROBABILITIES IN EVERYDAY SCENARIOS

Ex 83: Of all electric light bulbs produced, 5% are defective at manufacture. Six bulbs are randomly selected and tested. Find the probability that exactly two are defective (round to three decimal places).

$$P(\text{"2 defectives"}) \approx \boxed{0.031}$$

Answer:

- The random variable X represents the number of defective bulbs out of the 6 tested. Since each bulb is either defective (success, with probability 0.05) or not (failure, with probability 0.95), and the trials are independent, X follows a binomial distribution $B(6, 0.05)$.
- The probability of exactly 2 defective bulbs is calculated using the binomial formula:

$$\begin{aligned} P(\text{"2 defectives"}) &= P(X=2) \\ &= \binom{6}{2} 0.05^2 \, (1-0.05)^{6-2} \\ &\approx 0.0305... \\ &\approx 0.031 (\text{round to three decimal}) \end{aligned}$$

- Thus, the probability that exactly two bulbs are defective is approximately 0.031.

Ex 84: Weather forecasts indicate that on any given day in June, there is a 30% chance of rain. Over a week (7 days), the random variable X represents the number of rainy days. Find the probability that exactly 4 days are rainy (round to three decimal places).

$$P(\text{"4 rainy days"}) \approx \boxed{0.097}$$

Answer:

- The random variable X represents the number of rainy days out of 7, where each day is either rainy (success, with probability 0.3) or not (failure, with probability 0.7). Assuming independence across days, X follows a binomial distribution $B(7, 0.3)$.
- The probability of exactly 4 rainy days is calculated using the binomial formula:

$$\begin{aligned} P(\text{"4 rainy days"}) &= P(X=4) \\ &= \binom{7}{4} \cdot 0.3^4 \cdot (1-0.3)^{7-4} \\ &= 35 \cdot 0.0081 \cdot 0.343 \\ &\approx 0.097209 \\ &\approx 0.097 \text{ (rounded to three decimal places)} \end{aligned}$$

- Thus, the probability that exactly 4 days are rainy is approximately 0.097.

Ex 85: A student guesses on a 5-question true/false quiz, with a 50% chance of guessing each question correctly. The random variable X represents the number of correct answers. Find the probability that the student gets 1 or fewer questions correct (round to three decimal places).

$$P(\text{"1 or fewer correct"}) \approx \boxed{0.188}$$

Answer:

- The random variable X represents the number of correct answers out of 5, where each guess is either correct (success, with probability 0.5) or incorrect (failure, with probability 0.5). Assuming independent guesses, X follows a binomial distribution $B(5, 0.5)$.
- The probability of 1 or fewer correct answers is $P(X \leq 1)$, calculated using the binomial formula:

$$\begin{aligned} P(X \leq 1) &= P(X=0) + P(X=1) \\ &= \binom{5}{0} \cdot 0.5^0 \cdot (1-0.5)^{5-0} + \binom{5}{1} \cdot 0.5^1 \cdot (1-0.5)^{5-1} \\ &= 0.03125 + 0.15625 \\ &= 0.1875 \\ &\approx 0.188 \text{ (rounded to three decimal places)} \end{aligned}$$

- Thus, the probability that the student gets 1 or fewer questions correct is approximately 0.188.

Ex 86: A student sends party invitations to 4 friends via text message, and each friend has a 70% chance of accepting, independently of the others. The random variable X represents the number of friends who accept. Find the probability that all 4 friends accept (round to three decimal places).

$$P(\text{"only successes"}) \approx \boxed{0.240}$$

Answer:

- The random variable X represents the number of friends who accept out of 4, where each invitation is either accepted (success, with probability 0.7) or declined (failure, with probability 0.3). Assuming independence across invitations, X follows a binomial distribution $B(4, 0.7)$.
- The probability of "only successes" means all 4 friends accept, i.e., $P(X = 4)$, calculated using the binomial formula:

$$\begin{aligned} P(\text{"only successes"}) &= P(X = 4) \\ &= \binom{4}{4} \cdot 0.7^4 \cdot (1 - 0.7)^{4-4} \\ &= 1 \cdot 0.7^4 \cdot 0.3^0 \\ &= 1 \cdot 0.2401 \cdot 1 \\ &= 0.2401 \\ &\approx 0.240 \text{ (rounded to three decimal places)} \end{aligned}$$

- Thus, the probability that all 4 friends accept is approximately 0.240.

C.3.3 CALCULATING AN EXPECTED VALUE

Ex 87: A student guesses randomly on a 10-question multiple-choice quiz, where each question has 4 options and only one is correct. Find the average number of correct guesses the student makes.

$$\text{Average correct guesses} = \boxed{2.5}$$

Answer:

- The random variable X represents the number of correct guesses out of 10, where each guess is either correct (success, with probability $\frac{1}{4} = 0.25$, since there are 4 options) or incorrect (failure, with probability 0.75). Since the guesses are independent, X follows a binomial distribution $B(10, 0.25)$.
- The expected number of correct guesses is given by the mean of the binomial distribution:

$$E(X) = n \cdot p = 10 \cdot 0.25 = 2.5$$

- Thus, the average number of correct guesses is 2.5.

Ex 88: Weather forecasts indicate that on any given day in June, there is a 30% chance of rain. Find the average number of rainy days in a week (7 days).

$$\text{Average rainy days} = \boxed{2.1}$$

Answer:

- The random variable X represents the number of rainy days out of 7, where each day is either rainy (success, with probability 0.3) or not (failure, with probability 0.7). Since the days are independent, X follows a binomial distribution $B(7, 0.3)$.
- The expected number of rainy days is given by the mean of the binomial distribution:

$$E(X) = n \cdot p = 7 \cdot 0.3 = 2.1$$

- Thus, the average number of rainy days in a week is 2.1.

Ex 89: A delivery service has a 60% chance of delivering a package on time each day. Over 8 days, the random variable X represents the number of on-time deliveries. Find the average number of on-time deliveries over these 8 days.

$$\text{Average on-time deliveries} = \boxed{4.8}$$

Answer:

- The random variable X represents the number of on-time deliveries out of 8, where each delivery is either on time (success, with probability 0.6) or late (failure, with probability 0.4). Since the deliveries are independent, X follows a binomial distribution $B(8, 0.6)$.
- The expected number of on-time deliveries is given by the mean of the binomial distribution:

$$E(X) = n \cdot p = 8 \cdot 0.6 = 4.8$$

- Thus, the average number of on-time deliveries over 8 days is 4.8.

Ex 90: A basketball player is practicing 3-point shots. The probability that he successfully scores each shot is $\frac{4}{5}$, and each successful shot is worth 3 points. He takes 100 shots independently. Find the average number of points he scores after taking all 100 shots .

$$\text{Average points} = \boxed{240}$$

Answer:

- The random variable X represents the number of successful 3-point shots out of 100, where each shot is either a success (probability $\frac{4}{5}$) or a miss (probability $\frac{1}{5}$). Since the shots are independent, X follows a binomial distribution $B(100, \frac{4}{5})$.
- The expected number of successes (successful shots) is given by the mean of the binomial distribution: $E(X) = n \cdot p = 100 \cdot \frac{4}{5} = 80$.
- Each successful shot scores 3 points, so the average number of points is the expected number of successes multiplied by points per success:

$$\text{Average points} = E(X) \cdot 3 = 80 \cdot 3 = 240$$

- Thus, the average number of points scored after 100 shots is 240.