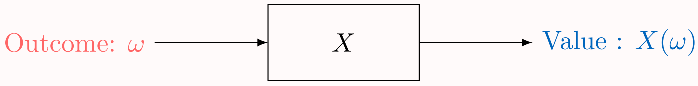
DISCRETE RANDOM VARIABLES

A RANDOM VARIABLES

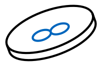
A.1 DEFINITIONS

Definition Random Variable

A **random variable**, denoted X, is a function that assigns a numerical value to each outcome ω in a random experiment. We write this value as $X(\omega)$.



The **possible values** of X are the real numbers that X can take.

Ex: Let X be the number of heads when tossing 2 fair coins:  (red coin) and  (blue coin). Find $X(H,T)$.

Answer: The outcome (H,T) means the red coin shows heads (H) and the blue coin shows tails (T). Since X counts heads, there's 1 head. Thus, $X(H,T) = 1$.

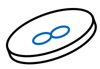
Definition Discrete Random Variable

A random variable is **discrete** if its set of possible values is finite or countably infinite. This means we can list all possible values.

Definition Events Involving a Random Variable

For a random variable X:

- $(X = x)$: The set of outcomes where X takes the value x.
- $(X \leq x)$: The set of outcomes where X is less than or equal to x.
- $(X \geq x)$: The set of outcomes where X is greater than or equal to x.

Ex: Let X be the number of heads when tossing 2 coins:  and . List the outcomes for $(X = 0)$, $(X = 1)$, $(X = 2)$, $(X \leq 1)$, and $(X \geq 1)$.

Answer:

- $(X = 0) = \{(T,T)\}$ (no heads).
- $(X = 1) = \{(T,H),(H,T)\}$ (one head).
- $(X = 2) = \{(H,H)\}$ (two heads).
- $(X \leq 1) = (X = 0) \cup (X = 1) = \{(T,T),(T,H),(H,T)\}$ (at most one head).
- $(X \geq 1) = (X = 1) \cup (X = 2) = \{(T,H),(H,T),(H,H)\}$ (at least one head).

A.2 PROBABILITY DISTRIBUTION

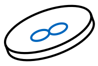
Definition Probability Distribution

The **probability distribution** of a random variable X lists the probability $P(X = x_i)$ for each possible value $x_1, x_2, \ldots, x_n$. It can be shown as a table or formula.

Proposition Characteristic of a Probability Distribution

For a random variable X with possible values $x_1, x_2, \ldots, x_n$, we have

- $0 \leq P(X = x_i) \leq 1$ for all $i = 1, \ldots, n$,
- $\displaystyle\sum_{i=1}^{n} P(X = x_i) = P(X = x_1) + P(X = x_2) + \cdots + P(X = x_n) = 1.$

Ex: Let X be the number of heads when tossing 2 fair coins:  and .

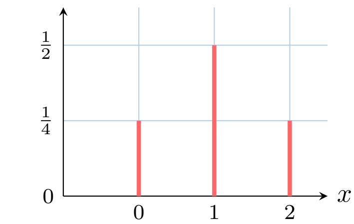
1. List the possible values of X.
2. Find the probability distribution.
3. Create the probability table.
4. Draw the probability distribution graph.

Answer:

1. Possible values: 0 (no heads), 1 (one head), 2 (two heads).
2. Probability distribution:
   - $P(X=0) = P(\{(T,T)\}) = \frac{1}{4}$,
   - $P(X=1) = P(\{(T,H),(H,T)\}) = \frac{2}{4} = \frac{1}{2}$,
   - $P(X=2) = P(\{(H,H)\}) = \frac{1}{4}$.
3. Probability table:

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

4. Graph:



A.3 EXISTENCE OF A RANDOM VARIABLE WITH A GIVEN PROBABILITY DISTRIBUTION

Usually, defining a random variable begins by establishing:

1. a sample space, that is, the set of all possible outcomes,
2. a probability associated with this sample space,
3. a function X that assigns a number to each outcome in the sample space.

This is quite a lengthy task. However, often, we prefer to directly define a random variable X with a given probability distribution, relying on the context of the situation being studied. For example, imagine we survey a class of 30 students about their siblings and obtain these results: 10 students have 0 siblings, 12 have 1 sibling, 5 have 2 siblings, and 3 have 3 siblings. We can then define the random variable X as the number of siblings of a randomly chosen student, with this probability distribution:

x	0	1	2	3
$P(X=x)$	$\frac{10}{30}$	$\frac{12}{30}$	$\frac{5}{30}$	$\frac{3}{30}$

The theorem below shows that it is always possible to construct a sample space, a probability, and a function X to obtain a random variable with this probability distribution.

Theorem Existence of a Random Variable with a Given Probability Distribution

Suppose you have possible values $x_1, x_2, \ldots, x_n$ and probabilities $p_1, p_2, \ldots, p_n$.
If:

- $0 \leq p_i \leq 1$ for each $i = 1, 2, \ldots, n$,
- $\sum_{i=1}^{n} p_i = p_1 + p_2 + \cdots + p_n = 1$,

then there exists a random variable X with the probability distribution $P(X = x_i) = p_i$ for each $i = 1, 2, \ldots, n$.

Method Defining a Random Variable X with a Valid Probability Distribution

In practice, we often define a random variable X directly by specifying its probability distribution. The key is to ensure that this distribution is valid, meaning it satisfies the conditions for a probability distribution: all probabilities must be non-negative and sum to 1.

Ex: We survey a class of 30 students about their siblings and obtain these results: 10 students have 0 siblings, 12 have 1 sibling, 5 have 2 siblings, and 3 have 3 siblings. We define a random variable X as the number of siblings of a randomly chosen student, with this probability distribution:

x	0	1	2	3
$P(X=x)$	$\frac{10}{30}$	$\frac{12}{30}$	$\frac{5}{30}$	$\frac{3}{30}$

Determine if this probability distribution is valid.

Answer:

- $P(X=x) \geq 0$ for all $x = 0, 1, 2, 3$ (true: $\frac{10}{30}$, $\frac{12}{30}$, $\frac{5}{30}$, and $\frac{3}{30}$ are all non-negative),
- $P(X=0) + P(X=1) + P(X=2) + P(X=3) = \frac{10}{30} + \frac{12}{30} + \frac{5}{30} + \frac{3}{30} = \frac{30}{30} = 1$ (true: the sum equals 1).

Since both conditions are satisfied, the probability distribution is valid.

B MEASURES OF CENTER AND SPREAD

B.1 EXPECTATION

The **expected value** of a random variable X is the "average you'd expect if you repeated the experiment many times". It's found by taking all possible values, multiplying each by its probability, and adding them up — essentially a weighted average where the probabilities act as the weights.

Definition Expected Value

For a random variable X with possible values $x_1, x_2, \ldots, x_n$, the **expected value**, $E(X)$, also called the **mean**, is:

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} x_i P(X = x_i) \\ &= x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_n P(X = x_n) \end{aligned}$$

Ex: You toss 2 fair coins, and X is the number of heads. The probability distribution is:

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Find the expected value of X.

Answer: Calculate $E(X)$ using the formula:

$$\begin{aligned} E(X) &= 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} \\ &= \frac{1}{2} + \frac{2}{4} \\ &= 1 \end{aligned}$$

So, on average, you expect 1 head when tossing 2 coins.

Proposition Linearity of Expectation

For any random variable X and constants a and b, the expected value of a linear transformation of X is:

$$E(aX + b) = aE(X) + b$$

This property is derived from two simpler rules:

- $E(aX) = aE(X)$ (The expectation of a scaled variable is the scaled expectation).
- $E(X + b) = E(X) + b$ (The expectation of a shifted variable is the shifted expectation).

Proof

The following derivation relies on the formula for the expectation of a function of a discrete random variable, $g(X)$, which is given by $E(g(X)) = \sum g(x_i)P(X = x_i)$.
Let the function be $g(X) = aX + b$.

$$\begin{aligned}
E(aX+b) &= \sum_i (ax_i + b)P(X = x_i) && \text{(by the formula for } E(g(X))\text{)} \\
&= \sum_i (ax_i P(X = x_i) + bP(X = x_i)) && \text{(distribute the probability)} \\
&= \sum_i ax_i P(X = x_i) + \sum_i bP(X = x_i) && \text{(split the summation)} \\
&= a\sum_i x_i P(X = x_i) + b\sum_i P(X = x_i) && \text{(factor out constants } a \text{ and } b\text{)} \\
&= aE(X) + b(1) && \text{(using } E(X) \text{ definition and } \textstyle\sum P(X = x_i) = 1\text{)} \\
&= aE(X) + b
\end{aligned}$$

B.2 VARIANCE AND STANDARD DEVIATION

The **variance** measures how spread out the values of a random variable are from its expected value. The **standard deviation** is the square root of the variance, giving a sense of typical deviation in the same units as X.

Definition Variance and Standard Deviation

The **variance**, denoted $V(X)$, is:

$$\begin{aligned}
V(X) &= \sum_{i=1}^{n} (x_i - E(X))^2 P(X = x_i) \\
&= (x_1 - E(X))^2 P(X = x_1) + (x_2 - E(X))^2 P(X = x_2) + \cdots + (x_n - E(X))^2 P(X = x_n)
\end{aligned}$$

The **standard deviation**, denoted $\sigma(X)$, is $\sigma(X) = \sqrt{V(X)}$.

Ex: You toss 2 fair coins, and X is the number of heads. The probability table is:

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Given $E(X) = 1$, find the variance.

Answer: Calculate $V(X)$:

$$\begin{aligned}
V(X) &= (0-1)^2 \times \frac{1}{4} + (1-1)^2 \times \frac{1}{2} + (2-1)^2 \times \frac{1}{4} \\
&= 1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} \\
&= \frac{1}{4} + 0 + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

The variance is $\frac{1}{2}$.

Proposition Computational Formula for Variance

A more convenient formula for computation is:

$$V(X) = E(X^2) - [E(X)]^2$$

Proof

Let $\mu = E(X)$.

$$\begin{aligned} V(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - E(2\mu X) + E(\mu^2) \quad \text{(by linearity of expectation)} \\ &= E(X^2) - 2\mu E(X) + \mu^2 \quad \text{(since } \mu \text{ and } \mu^2 \text{ are constants)} \\ &= E(X^2) - 2\mu(\mu) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

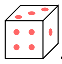
C CLASSICAL DISTRIBUTIONS

C.1 UNIFORM DISTRIBUTION

Definition Uniform Distribution

A random variable X follows a **uniform distribution** if each possible value has the same probability:

$$P(X = x) = \frac{1}{\text{Number of possible values}}, \quad \text{for any possible value } x$$

Ex: Let X be the result of rolling a fair die: .

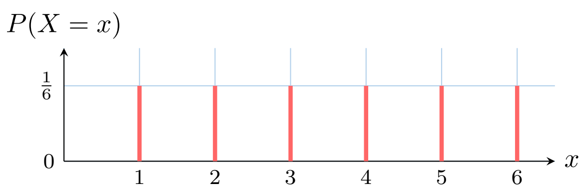
1. List the possible values of X.
2. Create the probability table.
3. Draw the probability distribution graph.

Answer:

1. Possible values: $1, 2, 3, 4, 5, 6$.
2. Probability table:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

3. Graph:



Proposition Expectation and Variance of a Uniform Distribution

For a random variable X that follows a uniform distribution on the set of integers $\{1, 2, \ldots, n\}$:

- The expected value is $E(X) = \frac{n+1}{2}$.
- The variance is $V(X) = \frac{n^2-1}{12}$.

Proof

Proof of the Expected Value $E(X)$: For a uniform distribution on $\{1, 2, \ldots, n\}$, the probability of any outcome is

$P(X = i) = \frac{1}{n}$.

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} i \cdot P(X = i) \\
&= \sum_{i=1}^{n} i \cdot \frac{1}{n} \\
&= \frac{1}{n} \sum_{i=1}^{n} i \quad \text{(factoring out the constant } 1/n\text{)} \\
&= \frac{1}{n} \left(\frac{n(n+1)}{2} \right) \quad \text{(using the formula for the sum of integers)} \\
&= \frac{n+1}{2}
\end{aligned}
$$

Ex: Let X be the random variable for the score on a roll of a fair six-sided die. Find the mean and variance of X.

Answer: The random variable X follows an uniform distribution on $\{1, 2, 3, 4, 5, 6\}$.

- $E(X) = \frac{6+1}{2} = 3.5$
- $V(X) = \frac{6^2-1}{12} = \frac{35}{12} \approx 2.92$

C.2 BERNOULLI DISTRIBUTION

A **Bernoulli distribution** models an experiment with two outcomes: success (1) or failure (0), like flipping a coin where heads is 1 and tails is 0. The probability of success is p.

Definition Bernoulli Distribution

A random variable X follows a **Bernoulli distribution** if:

- Possible values are 0 and 1.
- $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

We write $X \sim B(p)$.

Ex: A basketball player has an 80% chance of making a free throw. Let $X = 1$ if the shot is made, and $X = 0$ if it's missed.

1. Is X a Bernoulli random variable?
2. Find the probability of success.

Answer:

1. Yes, X has values 0 or 1, so it follows a Bernoulli distribution.
2. Probability of success: $P(X = 1) = 80\% = 0.8$.

Proposition Expectation and Variance of a Bernoulli Distribution

For a Bernoulli random variable X with a probability of success p, the following hold:

- The expected value is $E(X) = p$,
- The variance is $V(X) = p(1 - p)$,
- The standard deviation is $\sigma(X) = \sqrt{p(1-p)}$.

Proof

- $$
\begin{aligned}
E(X) &= 0 \times P(X = 0) + 1 \times P(X = 1) \\
&= 0 \times (1 - p) + 1 \times p \\
&= p
\end{aligned}
$$

- $V(X) = (0-p)^2(1-p) + (1-p)^2 p$
  $= p^2(1-p) + p(1-p)^2$
  $= p(1-p)[p + (1-p)]$
  $= p(1-p)$

C.3 BINOMIAL DISTRIBUTION

Suppose a basketball player takes n free throws, and we count the number of shots made. The probability of making a free throw is the same for each attempt, and each shot is independent of every other shot. This is an example of a binomial experiment.

Definition Binomial Experiment

A **binomial experiment** is a statistical experiment that consists of a sequence of repeated Bernoulli trials. It must satisfy the following four conditions:

1. **Fixed Number of Trials:** The experiment consists of a fixed number of trials, denoted by n.
2. **Independent Trials:** The outcome of each trial is independent of the outcomes of all other trials.
3. **Two Outcomes:** Each trial has only two possible outcomes, typically labeled "success" and "failure".
4. **Constant Probability:** The probability of success, denoted by p, is the same for each trial. The probability of failure is $1-p$.

A random variable X that counts the number of successes in a binomial experiment is called a **binomial random variable**.

Proposition Distribution of a Binomial Random Variable

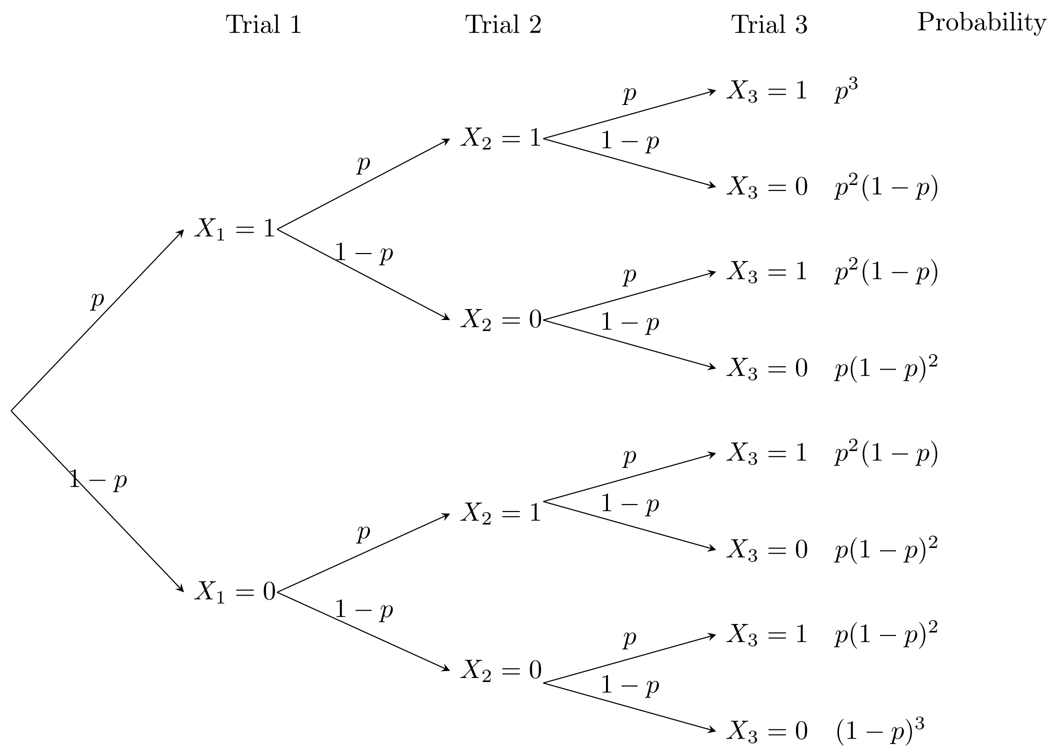
Let X be a binomial random variable with n independent trials and a probability of success p. The probability distribution of X is:

$$P(X = x) = \underbrace{\binom{n}{x}}_{\text{number of ways } x \text{ successes can be ordered among the } n \text{ trials}} \underbrace{p^x(1-p)^{n-x}}_{\text{probability of obtaining } x \text{ successes and } n-x \text{ failures in a particular order}} \quad \text{where} \quad x = 0, 1, 2, \ldots, n.$$

This is called the **binomial distribution**, and we write $X \sim B(n, p)$.

Proof

Consider the case where $n = 3$. Let X_1, X_2, and X_3 be three independent Bernoulli random variables, each with a probability of success p. Define $X = X_1 + X_2 + X_3$, which represents a binomial random variable.



- The possible values of X are $0, 1, 2, 3$.
- Probability calculations:
  - $P(X=0) = P(X_1 = 0 \text{ and } X_2 = 0 \text{ and } X_3 = 0)$
    $= P(X_1 = 0)P(X_2 = 0)P(X_3 = 0)$ (since X_1, X_2, X_3 are independent)
    $= (1-p)^3$
    $= \binom{3}{0} p^0 (1-p)^3$
  - $P(X=1) = P(X_1 = 1 \text{ and } X_2 = 0 \text{ and } X_3 = 0) + P(X_1 = 0 \text{ and } X_2 = 1 \text{ and } X_3 = 0)$
    $+ P(X_1 = 0 \text{ and } X_2 = 0 \text{ and } X_3 = 1)$
    $= p(1-p)^2 + p(1-p)^2 + p(1-p)^2$
    $= 3p(1-p)^2$
    $= \binom{3}{1} p^1 (1-p)^2$
  - $P(X=2) = P(X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 0) + P(X_1 = 1 \text{ and } X_2 = 0 \text{ and } X_3 = 1)$
    $+ P(X_1 = 0 \text{ and } X_2 = 1 \text{ and } X_3 = 1)$
    $= p^2(1-p) + p^2(1-p) + p^2(1-p)$
    $= 3p^2(1-p)$
    $= \binom{3}{2} p^2 (1-p)^1$
  - $P(X=3) = P(X_1 = 1 \text{ and } X_2 = 1 \text{ and } X_3 = 1)$
    $= p^3$
    $= \binom{3}{3} p^3 (1-p)^0$

Thus, $P(X = x) = \binom{3}{x} p^x (1-p)^{3-x}$ for $x = 0, 1, 2, 3$, matching the binomial distribution form.
The logic generalizes for any n. To obtain exactly x successes, we must choose x of the n trials to be successes, which can be done in $\binom{n}{x}$ ways. Each specific arrangement of x successes and $n-x$ failures has a probability of $p^x(1-p)^{n-x}$. By the addition rule, the total probability is the sum over all these arrangements, resulting in $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$.

Ex: A basketball player has an 80% chance of making a free throw and takes 5 shots. Let X be the number of shots made.

1. Is X a binomial random variable?

2. Find the probability of making 4 shots.

Answer:

1. Yes, X is a binomial random variable because it counts the number of successes (shots made) in 5 independent trials (free throws), each with a constant success probability of 0.8.
2. As $X \sim B(5, 0.8)$,

$$\begin{aligned} P(X = 4) &= \binom{5}{4}(0.8)^4(1-0.8)^1 \\ &= 5 \times 0.4096 \times 0.2 \\ &= 0.4096 \end{aligned}$$

The probability of making 4 shots is 0.4096.

Proposition Expectation and Variance of a Binomial Random Variable

For $X \sim B(n, p)$:

- $E(X) = np$ (expected value),
- $V(X) = np(1-p)$ (variance),
- $\sigma(X) = \sqrt{np(1-p)}$ (standard deviation).

Ex: A basketball player has an 80% chance of making a free throw and takes 5 shots. Find the mean and standard deviation of the number of successful shots.

Answer: Let X be the number of successful shots. Since each shot is independent and has a success probability of 0.8, we have $X \sim B(5, 0.8)$.

$$\begin{aligned} E(X) &= 5 \times 0.8 = 4, \\ V(X) &= 5 \times 0.8 \times (1-0.8) = 5 \times 0.8 \times 0.2 = 0.8, \\ \sigma(X) &= \sqrt{0.8} \approx 0.89. \end{aligned}$$

Mean is 4 successful shots, standard deviation is about 0.89.