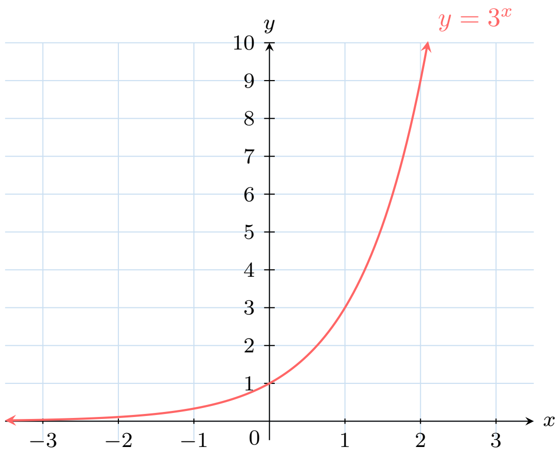
EXPONENTIAL FUNCTIONS

A EXPONENTIAL FUNCTION

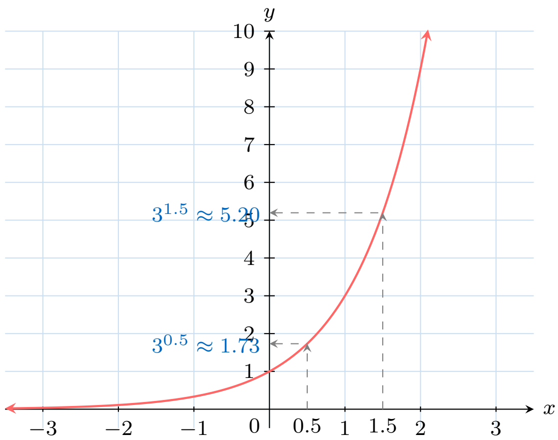
A.1 READING AND SKETCHING EXPONENTIAL FUNCTIONS

Ex 1: Find an approximation to the nearest integer, by reading the value from the graph of $f(x) = 3^x$:



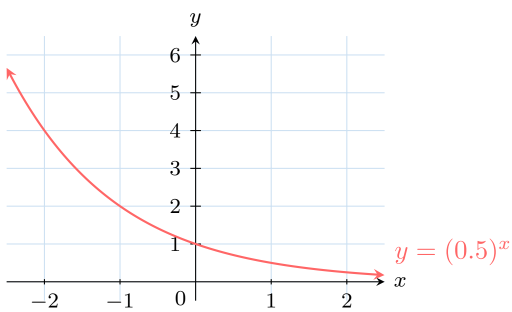
1. $\boxed{5} \leq 3^{1.5} < \boxed{6}$
2. $\boxed{1} \leq 3^{0.5} < \boxed{2}$

Answer: The values can be estimated visually using the graph above:



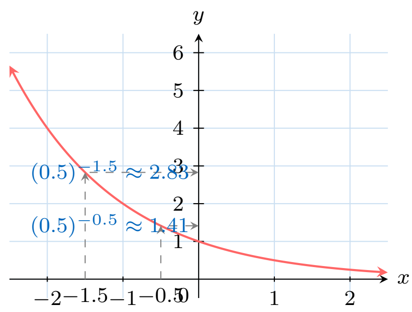
1. We read $3^{1.5} \approx 5.2$, so the answer is **between 5 and 6.**
$$\boxed{5} \leq 3^{1.5} < \boxed{6}$$
2. We read $3^{0.5} \approx 1.7$, so the answer is **between 1 and 2.**
$$\boxed{1} \leq 3^{0.5} < \boxed{2}$$

Ex 2: Find an approximation to the nearest integer, by reading the value from the graph of $f(x) = (0.5)^x$:



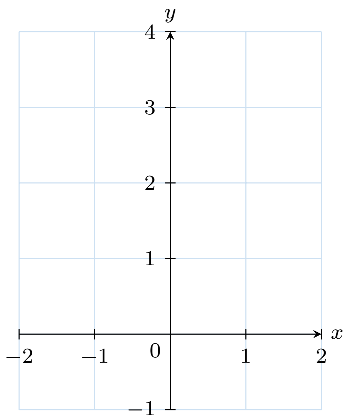
1. $\boxed{2} \leq (0.5)^{-0.5} < \boxed{3}$
2. $\boxed{1} \leq (0.5)^{-1.5} < \boxed{2}$

Answer: The values can be estimated visually using the graph above:



1. We read $(0.5)^{-0.5} \approx 1.41$, so the answer is **between 1 and 2.**
$$\boxed{1} \leq (0.5)^{-0.5} < \boxed{2}$$
2. We read $(0.5)^{-1.5} \approx 2.83$, so the answer is **between 2 and 3.**
$$\boxed{2} \leq (0.5)^{-1.5} < \boxed{3}$$

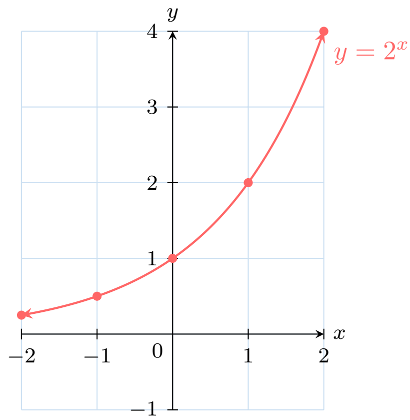
Ex 3: For the function $f(x) = 2^x$, sketch the graph of f.(You may fill in a table of values for $x = -2, -1, 0, 1, 2$.)



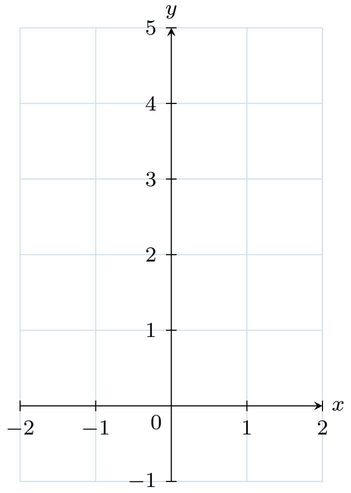
Answer: Fill in the table of values:

x	-2	-1	0	1	2
$f(x)$	0.25	0.5	1	2	4

Plot the points and draw the graph:



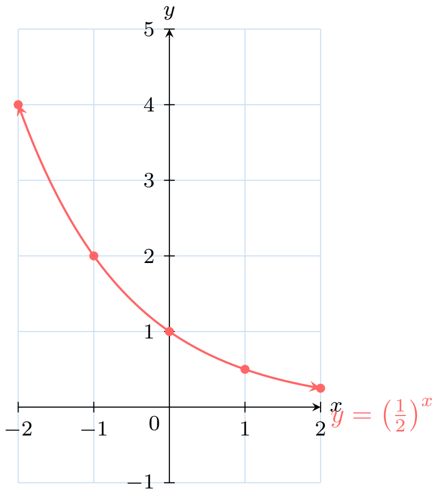
Ex 4: For the function $f(x) = \left(\frac{1}{2}\right)^x$, sketch the graph of f.(You may fill in a table of values for $x = -2, -1, 0, 1, 2$.)



Answer: Fill in the table of values:

x	-2	-1	0	1	2
$f(x)$	4	2	1	0.5	0.25

Plot the points and draw the graph:



A.2 EVALUATING EXPONENTIAL FUNCTIONS

Ex 5: For $f(x) = 3^x$, evaluate:

1. $f(2) = \boxed{9}$
2. $f(0) = \boxed{1}$
3. $f(-1) = \boxed{\frac{1}{3}}$

Answer:

1. $f(2) = 3^2$
   $= 9$
2. $f(0) = 3^0$
   $= 1$
3. $f(-1) = 3^{-1}$
   $= \frac{1}{3^1}$
   $= \frac{1}{3}$

Ex 6: For $f(x) = 10^x$, evaluate:

1. $f(2) = \boxed{100}$
2. $f(0) = \boxed{1}$
3. $f(-1) = \boxed{\frac{1}{10}}$

Answer:

1. $f(2) = 10^2$
   $= 100$
2. $f(0) = 10^0$
   $= 1$
3. $f(-1) = 10^{-1}$
   $= \frac{1}{10^1}$
   $= \frac{1}{10}$

Ex 7: For $f(x) = \left(\frac{1}{2}\right)^x$, evaluate:

1. $f(-2) = \boxed{4}$
2. $f(-1) = \boxed{2}$
3. $f(0) = \boxed{1}$
4. $f(1) = \boxed{\frac{1}{2}}$

Answer:

1. $f(-2) = \left(\frac{1}{2}\right)^{-2}$
   $= \left(\frac{2}{1}\right)^{2}$
   $= 2^2$
   $= 4$
2. $f(-1) = \left(\frac{1}{2}\right)^{-1}$
   $= \left(\frac{2}{1}\right)^{1}$
   $= 2$
3. $f(0) = \left(\frac{1}{2}\right)^{0}$
   $= 1$
4. $f(1) = \left(\frac{1}{2}\right)^{1}$
   $= \frac{1}{2}$

B EXPONENTIAL VS. LINEAR RELATIONSHIPS

B.1 RECOGNIZING LINEAR, EXPONENTIAL, OR NEITHER RELATIONSHIPS FROM TABLES

MCQ 8:

x	0	1	2	3
y	1	5	25	125

What is the relationship between the two variables?

☐ linear relationship

☒ exponential relationship

☐ neither

Answer: For each increase of 1 in x, the value of y is multiplied by 5.

$$x : 0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3$$
$$y : 1 \xrightarrow{\times 5} 5 \xrightarrow{\times 5} 25 \xrightarrow{\times 5} 125$$

So, this relationship is **exponential** ($y = 5^x$).

MCQ 9:

x	0	1	2	3
y	0	20	40	60

What is the relationship between the two variables?

☒ linear relationship

☐ exponential relationship

☐ neither

Answer: For each increase of 1 in x, the value of y increases by 20.

$$x : 0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3$$
$$y : 0 \xrightarrow{+20} 20 \xrightarrow{+20} 40 \xrightarrow{+20} 60$$

So, this relationship is **linear** ($y = 20x$).

MCQ 10:

x	0	2	4	6
y	1	6	11	16

What is the relationship between the two variables?

☒ linear relationship

☐ exponential relationship

☐ neither

Answer: For each increase of 2 in x, the value of y increases by 5.

$$x : 0 \xrightarrow{+2} 2 \xrightarrow{+2} 4 \xrightarrow{+2} 6$$
$$y : 1 \xrightarrow{+5} 6 \xrightarrow{+5} 11 \xrightarrow{+5} 16$$

So, this relationship is **linear** ($y = 1 + 2.5x$).

MCQ 11:

x	0	1	2	3
y	3	6	12	24

What is the relationship between the two variables?

☐ linear relationship

☒ exponential relationship

☐ neither

Answer: For each increase of 1 in x, the value of y is multiplied by 2.

$$x : 0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3$$
$$y : 3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24$$

So, this relationship is **exponential** ($y = 3 \times 2^x$).

MCQ 12:

x	0	1	2	3
y	2	5	10	17

What is the relationship between the two variables?

☐ linear relationship

☐ exponential relationship

☒ neither

Answer: The differences between the y-values are not constant:

$$y : 2 \xrightarrow{+3} 5 \xrightarrow{+5} 10 \xrightarrow{+7} 17$$

The ratios between y-values are also not constant:

$$\frac{5}{2} = 2.5, \quad \frac{10}{5} = 2, \quad \frac{17}{10} = 1.7$$

So, this relationship is **neither linear nor exponential**.

B.2 RECOGNIZING LINEAR AND EXPONENTIAL RELATIONSHIPS IN REAL-LIFE CONTEXTS

MCQ 13: The number of infected people with Covid doubles each day. What is the relationship between the two variables (day and number of infected)?

☐ linear relationship

☒ exponential relationship

☐ neither

Answer: For each increase of 1 in the number of days, the number of infected people is multiplied by 2. So, this relationship is **exponential**.

MCQ 14: A bus ticket costs \$2, plus an extra \$0.50 for each additional zone crossed. What is the relationship between the number of zones crossed and the total price?

☒ linear relationship

☐ exponential relationship

☐ neither

Answer: For each increase of 1 in the number of zones, the total price increases by \$0.50. So, this relationship is **linear**.

MCQ 15: The amount of money in a bank account increases by 5% each year due to compounded interest. What is the relationship between the two variables (year and amount)?

- ☐ linear relationship
- ☒ exponential relationship
- ☐ neither

Answer: For each increase of 1 in the number of years, the amount is multiplied by 1.05. So, this relationship is **exponential**.

MCQ 16: A cyclist travels at a constant speed of 15 km per hour. What is the relationship between the number of hours and the distance traveled?

- ☒ linear relationship
- ☐ exponential relationship
- ☐ neither

Answer: For each increase of 1 in the number of hours, the distance increases by 15 km. So, this relationship is **linear**.

C EXPONENTIAL MODELS

C.1 MODELING REAL-WORLD SITUATIONS WITH EXPONENTIAL FUNCTIONS

Ex 17: A population of bacteria doubles every second. At time $x = 0$, there is a single bacterium.
Find the function to model this growth.

$$P(x) = \boxed{2^x}$$

Answer: Let $P(x)$ be the population of bacteria after x seconds. We have:

$$\begin{aligned} P(0) &= 1 = 2^0 \\ P(1) &= 2 = 2^1 \\ P(2) &= 4 = 2^2 \\ &\dots \\ P(x) &= 2^x \end{aligned}$$

So, the population after x seconds is $P(x) = 2^x$.

Ex 18: A species of bear is introduced to a large island off Alaska where previously there were no bears. 6 pairs of bears were introduced in 1998. It is expected that the population will increase according to $B(t) = B_0 \times (1.13)^t$ where t is the time, in years, since the introduction.

1. Find B_0.

   $\boxed{12}$ bears

2. Find the expected bear population in 2018.

   $\boxed{138}$ bear (round to the nearest integer)

3. Find the expected percentage increase in population from 1998 to 2018.

   $\boxed{1050}$% (round to the nearest integer)

Answer:

1. $B_0 = 6$ pairs $= 12$ bears.
2. 2018 is 20 years after 1998, so $t = 20$.

$$\begin{aligned} B(20) &= 12 \times (1.13)^{20} \\ &\approx 12 \times 11.495 \\ &\approx 137.94 \\ &\approx 138 \text{ bears} \end{aligned}$$

3. The expected percentage increase is

$$\frac{138 - 12}{12} \times 100\% = 1050\%$$

Ex 19: Sarah buys a piece of artwork for \$1500 that is expected to appreciate (increase in value) by 8% each year.

1. Determine a model for A_n, the value of the artwork after n years.

   $$A_n = \boxed{1500 \times (1.08)^n}$$

2. Is this an example of exponential growth?
   $\boxed{Yes}$

3. Calculate the estimated value of the artwork in 6 years' time.

   \$ $\boxed{2380}$ (round to the nearest integer)

Answer:

1. Initial value $A_0 = \$1500$, annual growth rate $r = 8\%$.The model is:

$$\begin{aligned} A_0 &= 1500 \\ A_1 &= 1500 \times 1.08 \\ A_2 &= 1500 \times (1.08)^2 \\ A_3 &= 1500 \times (1.08)^3 \\ &\vdots \\ A_n &= 1500 \times (1.08)^n \end{aligned}$$

   So, $A_n = 1500 \times (1.08)^n$.

2. Yes, this is an example of exponential growth because the value is multiplied by the same factor (1.08) each year.
3. Substitute $n = 6$:

$$\begin{aligned} A_6 &= 1500 \times (1.08)^6 \\ &\approx 1500 \times 1.586874 \\ &\approx 2380 \end{aligned}$$

   The estimated value in 6 years is \$2 380 (rounded to the nearest integer).

Ex 20: Maxime has an Uncle Scrooge coin worth \$500. Each year, the coin's value increases by 20%.

1. Determine a model for C_n, the value of the coin after n years.

$$C_n = \boxed{500 \times (1.20)^n}$$

2. Is this an example of exponential growth?
$\boxed{Yes}$

3. Calculate the estimated value of the coin in 6 years' time.

$$\$\boxed{1493} \text{ (round to the nearest integer)}$$

Answer:

1. Initial value $C_0 = \$500$, annual growth rate $r = 20\%$.The model is:
$$\begin{aligned} C_0 &= 500 \\ C_1 &= 500 \times 1.20 \\ C_2 &= 500 \times (1.20)^2 \\ C_3 &= 500 \times (1.20)^3 \\ &\vdots \\ C_n &= 500 \times (1.20)^n \end{aligned}$$
So, $C_n = 500 \times (1.20)^n$.

2. Yes, this is an example of exponential growth because the value is multiplied by the same factor (1.20) each year.

3. Substitute $n = 6$:
$$\begin{aligned} C_6 &= 500 \times (1.20)^6 \\ &\approx 500 \times 2.985984 \\ &\approx 1493 \end{aligned}$$
The estimated value in 6 years is \$1490 (rounded to the nearest integer).

Ex 21: A certain radioactive substance loses 12% of its mass each year. Initially, the sample weighs 200 g.

1. Determine a model for M_n, the mass (in grams) remaining after n years.

$$M_n = \boxed{200 \times (0.88)^n}$$

2. Is this an example of exponential decay?
$\boxed{Yes}$

3. Calculate the mass remaining after 10 years.

$$\boxed{56} \text{ g (round to the nearest integer)}$$

Answer:

1. Initial mass $M_0 = 200$ g, annual loss rate $= 12\%$.The model is:
$$\begin{aligned} M_0 &= 200 \\ M_1 &= 200 \times 0.88 \\ M_2 &= 200 \times (0.88)^2 \\ M_3 &= 200 \times (0.88)^3 \\ &\vdots \\ M_n &= 200 \times (0.88)^n \end{aligned}$$
So, $M_n = 200 \times (0.88)^n$.

2. Yes, this is an example of exponential decay because the mass is multiplied by the same factor (0.88) each year.

3. Substitute $n = 10$:
$$\begin{aligned} M_{10} &= 200 \times (0.88)^{10} \\ &\approx 200 \times 0.2886 \\ &\approx 56 \end{aligned}$$
So, the mass remaining after 10 years is 56 g (rounded to the nearest integer).