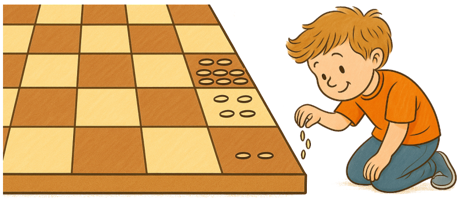
EXPONENTS

Exponents are an efficient way of expressing repeated multiplication, and they help us work with large numbers more easily.

A POSITIVE EXPONENTS

Discover: Imagine you have a chessboard. You place two grains of wheat on the first square, four grains on the second square, eight grains on the third square, and so on, doubling the number of grains on each next square.



How many grains of wheat are on the last square of a chessboard with 64 squares?

Answer:

Square number	Grain number
1	2
2	2×2
3	$2\times 2\times 2$
$\vdots$	$\vdots$
64	$\overbrace{2\times 2\times\cdots\times 2}^{64 \text{ factors}}$

Rather than writing $\overbrace{2\times 2\times\cdots\times 2}^{64 \text{ factors}}$, we can write this product as 2^{64}. Using a calculator, $2^{64} = 18\,446\,744\,073\,709\,551\,616$ which is over 2,000 times the annual world production of wheat, which in the period 2020-21 was an estimated 800 million metric tonnes.

Definition Exponentiation

Exponentiation is repeated multiplication of a number by itself:

$$\underset{\text{base}}{} \to \overset{\text{exponent}\downarrow}{2^4} = \overbrace{2\times 2\times 2\times 2}^{4 \text{ factors}}$$

Ex: Write using exponent notation: $5\times 5\times 5$

Answer: $5\times 5\times 5 = 5^3$

Definition Vocabulary

Value	Expanded form	Exponent notation	Spoken form
2	2	2^1	2 or 2 raised to the power 1
4	2×2	2^2	2 squared or 2 raised to the power 2
8	$2\times 2\times 2$	2^3	2 cubed or 2 raised to the power 3
16	$2\times 2\times 2\times 2$	2^4	2 raised to the power 4
32	$2\times 2\times 2\times 2\times 2$	2^5	2 raised to the power 5

Ex: Find the value for 2^3.

Answer:

$$\begin{aligned} 2^3 &= 2\times 2\times 2 \\ &= 8 \end{aligned}$$

B NEGATIVE EXPONENTS

Discover: To understand negative exponents, let's explore the pattern of multiplying by 2:

$$\begin{array}{ccccc} 2 & \xrightarrow{\times 2} & 2\times 2 & \xrightarrow{\times 2} & 2\times 2\times 2 \\ 2^1 & \xrightarrow{+1} & 2^2 & \xrightarrow{+1} & 2^3 \end{array}$$

From this pattern, you can see:

- $2^1 = 2$
- $2^2 = 2 \times 2$
- $2^3 = 2 \times 2 \times 2$

Because the reverse operation of multiplication is division, we can divide by 2 repeatedly to extend the pattern:

$$\begin{array}{ccccccccccccc} \frac{1}{2\times 2\times 2} & \xleftarrow{\div 2} & \frac{1}{2\times 2} & \xleftarrow{\div 2} & \frac{1}{2} & \xleftarrow{\div 2} & 1 & \xleftarrow{\div 2} & 2 & \xrightarrow{\times 2} & 2\times 2 & \xrightarrow{\times 2} & 2\times 2\times 2 \\ 2^{-3} & \xleftarrow{-1} & 2^{-2} & \xleftarrow{-1} & 2^{-1} & \xleftarrow{-1} & 2^0 & \xleftarrow{-1} & 2^1 & \xrightarrow{+1} & 2^2 & \xrightarrow{+1} & 2^3 \end{array}$$

From this pattern, you can see:

- $2^0 = 1$
- $2^{-1} = \dfrac{1}{2}$
- $2^{-2} = \dfrac{1}{2 \times 2}$
- $2^{-3} = \dfrac{1}{2 \times 2 \times 2}$

Definition Exponentiation for a negative exponent

For a negative exponent, exponentiation is repeated division:

$$a^{-n} = \frac{1}{\underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}} = \frac{1}{a^n} \qquad \text{and} \qquad a^0 = 1$$

In particular, $a^{-1} = \dfrac{1}{a}$.

Ex: Write 3^{-2} as a fraction.

Answer:

$$3^{-2} = \frac{1}{3 \times 3} = \frac{1}{9}$$

C EXPONENT LAW 1

Discover:

$$7^3 \times 7^2 = \overbrace{7 \times 7 \times 7}^{3 \text{ factors}} \times \overbrace{7 \times 7}^{2 \text{ factors}} = \overbrace{7 \times 7 \times 7 \times 7 \times 7}^{3+2 \text{ factors}} = 7^{3+2}$$

In general, when a number a is raised to the power m and multiplied by the same number raised to the power n, that is,

$$a^m \times a^n,$$

the result is equal to a raised to the sum of the exponents:

$$a^m \times a^n = a^{m+n}.$$

Proposition Exponent law 1

$$a^m \times a^n = a^{m+n}$$

Proof

$$\begin{aligned} a^m \times a^n &= \overbrace{a \times \cdots \times a}^{m \text{ factors}} \times \overbrace{a \times \cdots \times a}^{n \text{ factors}} \\ &= \overbrace{a \times \cdots \times a}^{m+n \text{ factors}} \\ &= a^{m+n} \end{aligned}$$

Ex: Simplify $5^2 \times 5^4$.

Answer:

$$\begin{aligned} 5^2 \times 5^4 &= 5^{2+4} \\ &= 5^6 \end{aligned}$$

D EXPONENT LAW 2

Discover: Let's look at an example:

$$\begin{aligned} \frac{7^5}{7^2} &= \frac{\overbrace{\cancel{7} \times \cancel{7} \times 7 \times 7 \times 7}^{5 \text{ factors}}}{\underbrace{\cancel{7} \times \cancel{7}}_{2 \text{ factors}}} \\ &= \overbrace{7 \times 7 \times 7}^{5-2 \text{ factors}} \\ &= 7^{5-2} \end{aligned}$$

In general, when a number a is raised to the power m and divided by the same number raised to the power n, that is,

$$\frac{a^m}{a^n}$$

the result is a raised to the difference of the exponents:

$$a^{m-n}$$

Proposition Exponent Law 2

$$\frac{a^m}{a^n} = a^{m-n}$$

Ex: Simplify $\frac{5^7}{5^3}$.

Answer:

$$\begin{aligned} \frac{5^7}{5^3} &= 5^{7-3} \\ &= 5^4 \end{aligned}$$

E EXPONENT LAW 3

Discover: Let's look at an example:

$$\begin{aligned} \left(5^2\right)^3 &= (\overbrace{5 \times 5}^{2 \text{ factors}})^3 \\ &= \overbrace{(\overbrace{5 \times 5}^{2 \text{ factors}}) \times (\overbrace{5 \times 5}^{2 \text{ factors}}) \times (\overbrace{5 \times 5}^{2 \text{ factors}})}^{3 \text{ factors}} \\ &= 5^{2+2+2} \\ &= 5^{2 \times 3} \end{aligned}$$

In general, when a number a is raised to the power m, and that result is raised to the power n, that is,

$$(a^m)^n,$$

the result is a raised to the product of the exponents:

$$a^{m\times n}$$

Proposition Exponent Law 3

$$(a^m)^n = a^{m\times n}$$

Ex: Simplify $(5^2)^5$.

Answer:

$$\begin{aligned}(5^2)^5 &= 5^{2\times 5}\\ &= 5^{10}\end{aligned}$$

F EXPONENT LAW 4

Discover: Let's look at an example:

$$\begin{aligned}(3\times 5)^2 &= (3\times 5)\times(3\times 5)\\ &= 3\times 5\times 3\times 5\\ &= (3\times 3)\times(5\times 5)\\ &= 3^2\,5^2\end{aligned}$$

In general, when you multiply two numbers a and b, and then raise the product to the power n, that is,

$$(ab)^n,$$

the result is each factor raised to the power n:

$$(ab)^n = a^n\,b^n$$

Proposition Exponent Law 4

$$(ab)^n = a^n\,b^n$$

Ex: Simplify $(2\times 5)^3$.

Answer:

$$(2\times 5)^3 = 2^3\,5^3$$

G EXPONENT LAW 5

Discover: Let's look at an example:

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \left(\frac{5}{3}\right)\times\left(\frac{5}{3}\right)\\ &= \frac{5\times 5}{3\times 3}\\ &= \frac{5^2}{3^2}\end{aligned}$$

In general, when a quotient $\frac{a}{b}$ is raised to a power n, that is,

$$\left(\frac{a}{b}\right)^n,$$

the result is the numerator raised to that power divided by the denominator raised to that power:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Proposition Exponent Law 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Ex: Calculate $\left(\frac{5}{3}\right)^2$.

Answer:

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \frac{5^2}{3^2} \\ &= \frac{25}{9}\end{aligned}$$

H ORDER OF OPERATIONS

The order of operations is a set of guidelines that help us solve mathematical expressions in a consistent manner.

Definition Order of Operations

To solve mathematical expressions accurately, we follow the **order of operations**, which is commonly remembered using the acronym **PEMDAS**:

1. P: Parentheses
2. E: Exponents
3. M: Multiplication
4. D: Division
5. A: Addition
6. S: Subtraction

The order of operations proceeds from top to bottom, meaning we start with parentheses, then exponents, and so on. However, multiplication and division, as well as addition and subtraction, have the same level of priority. In these cases, we work from left to right.

Ex: Evaluate $(1+2)\times 2^3+4$

Answer:

$$\begin{aligned}(1+2)\times 2^3+4 &= (1+2)\times(2^3+4 && \text{(parentheses: } (1+2)=3)\\ &= 3\times 2^3+4 && \text{(exponent: } 2^3=8)\\ &= 3\times 8+4 && \text{(multiplication: } 3\times 8=24)\\ &= 24+4 && \text{(addition: } 24+4=28\\ &= 28) \end{aligned}$$