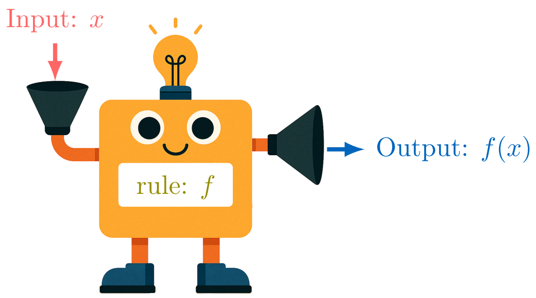
FUNCTIONS

A DEFINITIONS

Discover: A function is like a machine that produces an output from an input according to a rule.

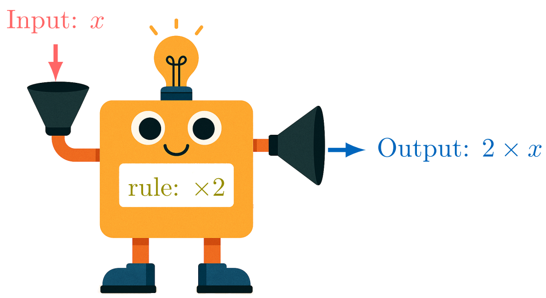


To represent this machine, we write $f(\text{input}) = \text{output}$. The brackets ( ) indicate the action of the function f on the input. We use function notation to name functions and their variables, replacing "input" by "x" and "output" by "$f(x)$".
We can write this function as

$$f : x \mapsto f(x)$$

function f such that x is transformed to $f(x)$

For example, if the rule is "twice the input":



we have $f(x) = 2x$:

$$f : x \mapsto 2x$$

function f such that x is transformed to $2x$

When the input is $x = 1$, we get:

$$\begin{aligned} f(1) &= 2 \times (1) \\ &= 2 \end{aligned}$$

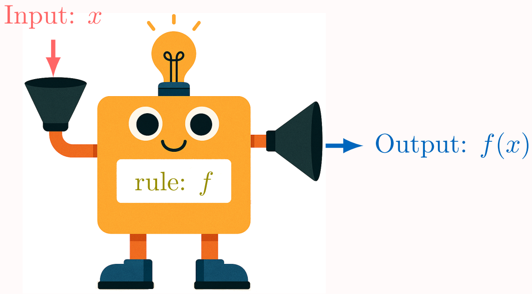
The table of values below shows the output values for different input values:

Input: x	0	1	2
Output: $f(x)$	0	2	4

Twice the input

Definition Function

From an input value x, a **function** f produces an output value $f(x)$.
We can write:



$$f : x \mapsto f(x)$$

function f such that x is transformed to $f(x)$

- $f(x)$ is read as "f of x".
- $f(x)$ is called the **image** of x.

Ex: For $f(x) = 2x - 1$ (the function that doubles the input and subtracts 1), find $f(5)$.

Answer: $f(5) = 2 \times (5) - 1 \quad$ (substituting x by (5))
$= 9$

B TABLES OF VALUES

Definition Table of Values

The **table of values** for a function f provides a listing of pairs $(x, f(x))$, where x is an input value and $f(x)$ is the corresponding output value produced by the function f.

Ex: For $f(x) = x^2$, complete the following table:

x	-2	-1	0	1	2
$f(x)$					

Answer:

- $f(-2) = (-2)^2 \quad$ (substituting x by (-2))
  $= 4$
- $f(-1) = (-1)^2 \quad$ (substituting x by (-1))
  $= 1$
- $f(0) = (0)^2 \quad$ (substituting x by (0))
  $= 0$
- $f(1) = (1)^2 \quad$ (substituting x by (1))
  $= 1$
- $f(2) = (2)^2 \quad$ (substituting x by (2))
  $= 4$

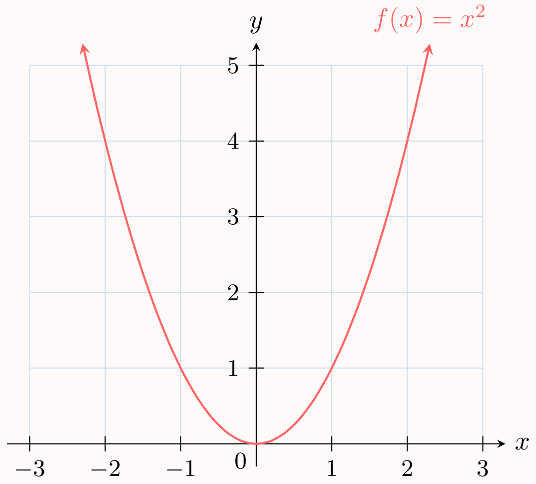
So the completed table is:

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4

C GRAPHS

Definition Graph

A **graph** of a function is the set of all points $(x, f(x))$ in the plane, where x is an input and $f(x)$ is its output.



Method Plotting a Line Graph from a Table

To plot the graph of a function from a table of values, follow these steps:

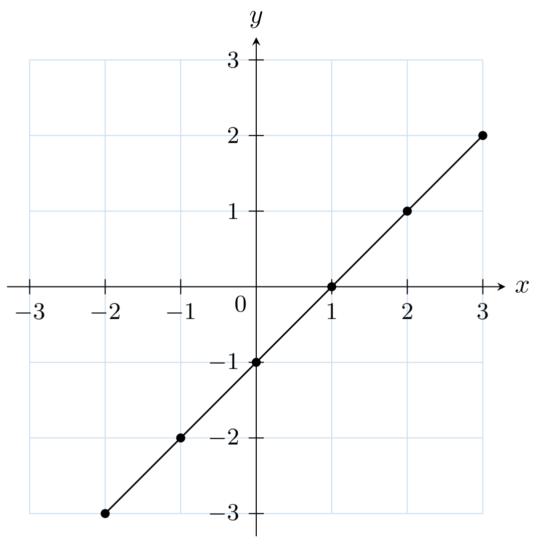
- Plot each point $(x, f(x))$ from the table onto the coordinate plane.
- Connect the points with straight line segments.

Ex: Here is a table of values for the function $f(x) = x - 1$:

x	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	0	1	2

Plot the line graph of f.

Answer: Plot the points $(-2, -3)$, $(-1, -2)$, $(0, -1)$, $(1, 0)$, $(2, 1)$, and $(3, 2)$. Then, connect the points with straight segments to form the line graph.

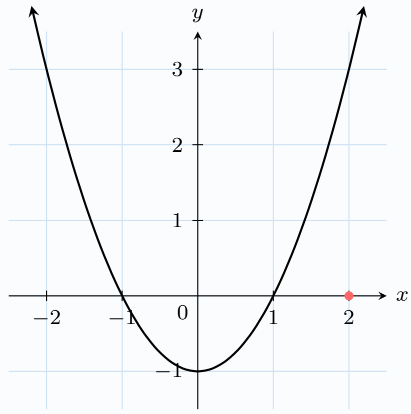


D READING VALUES AND SOLVING $f(x) = y$ ON A GRAPH

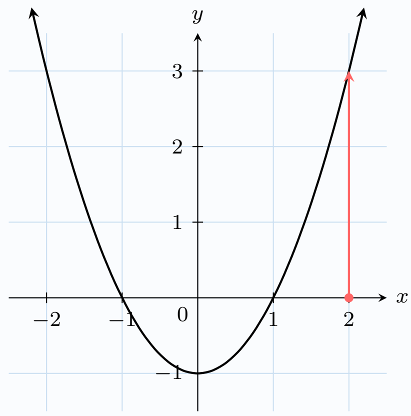
Method Finding the value $f(x)$ using a graph

To find $f(2)$ on a graph, follow these steps:

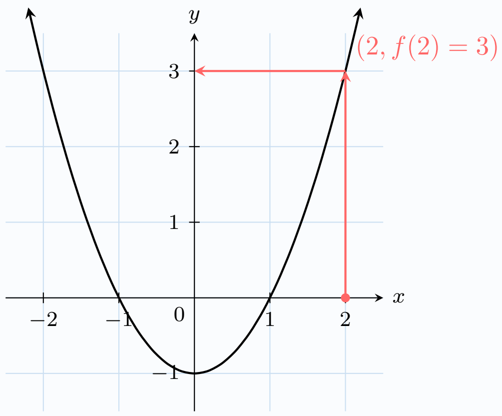
1. **Locate the x-value:** Find $x = 2$ on the x-axis.



2. **Move vertically to the curve:** From $x = 2$, draw a vertical line up to the graph.



3. **Read the y-value:** At the intersection with the curve, move horizontally to the y-axis to find the value $f(2)$.

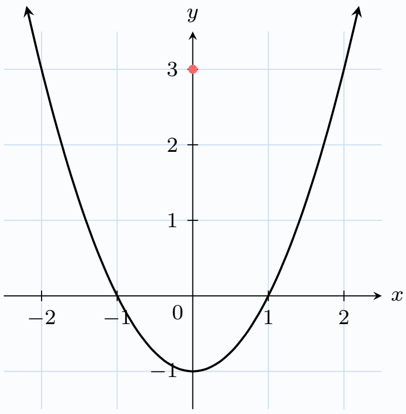


Thus, $f(2) = 3$.

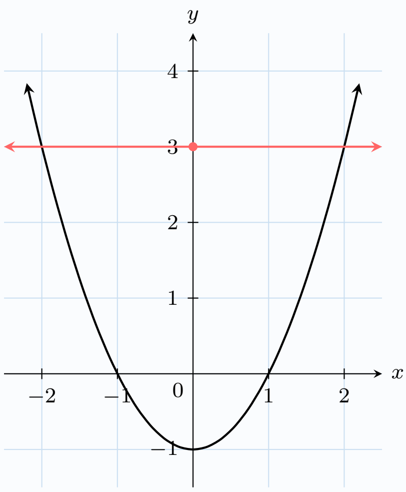
Method **Finding x such that $f(x) = y$ using a graph**

To find x where $f(x) = 3$ on this graph:

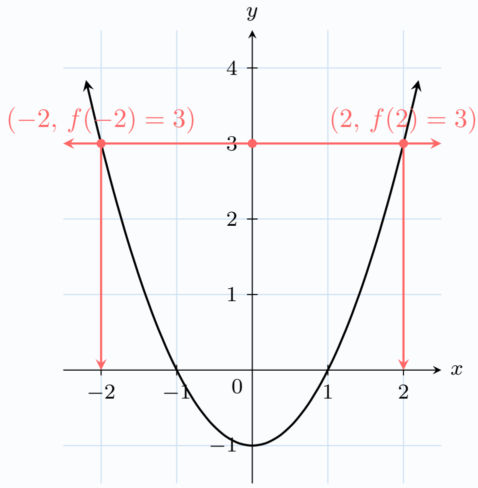
1. **Locate the y-value on the y-axis:** Find 3 on the y-axis.



2. **Draw horizontally to the graph of the function:** Draw a horizontal line from $y = 3$ to the curve.



3. **Read the x-values:** From the intersection points, draw vertical lines down to the x-axis and read the corresponding x-values.



Thus, the values of x for which $f(x) = 3$ are $x = 2$ and $x = -2$.

E SOLVING $f(x) = y$ ALGEBRAICALLY

Method Solving $f(x) = y$ algebraically

To find x such that $f(x) = y$:

- Write the equation $f(x) = y$.
- Solve for x using algebraic methods (e.g., inverse operations, isolating x).

Ex: Let $f(x) = 3x + 12$. Find all x such that $f(x) = 0$.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 0 \\ 3x + 12 &= 0 \\ 3x + 12 - 12 &= 0 - 12 && \text{(subtract 12 from both sides)} \\ 3x &= -12 \\ \frac{3x}{3} &= \frac{-12}{3} && \text{(divide both sides by 3)} \\ x &= -4 \end{aligned}$$

So the solution is $x = -4$.
We can check this by calculating $f(-4)$:

$$f(-4) = 3 \times (-4) + 12 = -12 + 12 = 0$$

So $f(-4) = 0$, as expected.

F DOMAIN

Definition Domain

Given a function, the **domain** is the set of possible x values for which the function is defined.

Method Finding the Domain

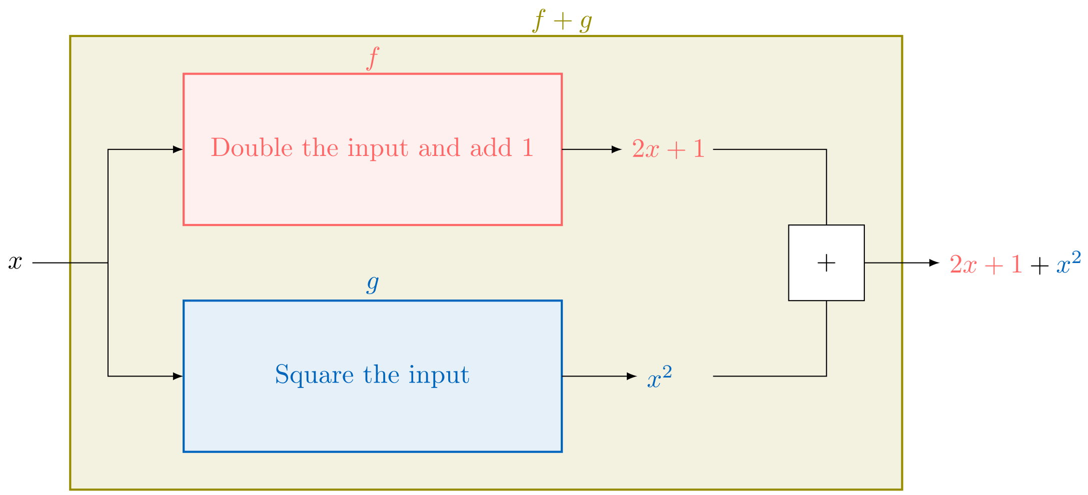
To find the domain of a function, identify the values of x that make the function undefined (such as division by zero or taking the square root of a negative number), and exclude them from the set of all real numbers.

Ex: Find the domain of the function $f(x) = \sqrt{x}$.

Answer: The square root is undefined for negative numbers ($\sqrt{-1}$ is undefined). So, the domain is $[0, +\infty)$.

G ALGEBRA OF FUNCTIONS

Discover: Let $f(x) = 2x + 1$ and $g(x) = x^2$.$f(x) + g(x)$ means that for a given input x, you calculate $f(x)$ and $g(x)$ separately, then add the results.This is illustrated by the function machine below:



So,

$$\begin{aligned} f(x) + g(x) &= f(x) + g(x) \\ &= 2x + 1 + x^2 \end{aligned}$$

Definition Operations on Functions

Given functions f and g:

- The **sum of functions** is $x \mapsto f(x) + g(x)$.
- The **product of functions** is $x \mapsto f(x) \times g(x)$.

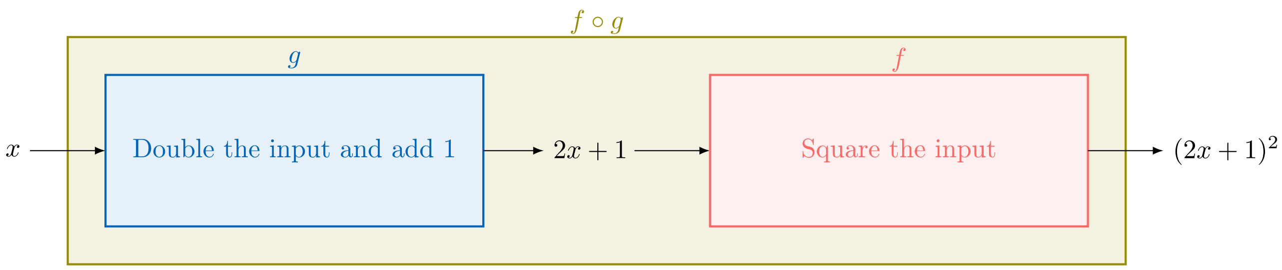
Ex: Let $f(x) = 2x + 1$ and $g(x) = x^4 - 1$. Find $f(x) + g(x)$.

Answer:

$$\begin{aligned} f(x) + g(x) &= (2x+1) + (x^4 - 1) \\ &= x^4 + 2x \end{aligned}$$

H COMPOSITION

Discover: Let $f(x) = x^2$ and $g(x) = 2x + 1$.$f(g(x))$ means that you first apply g to x, and then apply f to the result. In other words, x goes through g (becomes $2x + 1$), and then f (becomes $(2x + 1)^2$).This is illustrated by the function machine below:



Algebraically,

$$\begin{aligned} f(g(x)) &= f\big(g(x)\big) \\ &= f(2x+1) \\ &= (2x+1)^2 \end{aligned}$$

Definition Composition of Functions

Given two functions f and g, the **composition of functions** of f and g is $x \mapsto f(g(x))$.
This means you first apply g to x, then apply f to the result.

Ex: Let $f(x) = x^2$ and $g(x) = 2x + 1$.

1. Find $f(g(x))$.
2. Find $g(f(x))$.
3. Is $f(g(x)) = g(f(x))$?

Answer:

1. $$\begin{aligned} f(g(x)) &= f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2 \\ &= 4x^2 + 4x + 1 \end{aligned}$$
2. $$\begin{aligned} g(f(x)) &= g(f(x)) \\ &= g(x^2) \\ &= 2x^2 + 1 \end{aligned}$$
3. So $f(g(x)) \neq g(f(x))$.

I INVERSE FUNCTION

The operations of $+$ and $-$, and $\times$ and $\div$, are called **inverse operations** because one "undoes" the other.
In a similar way, some functions have **inverse functions** that "undo" the effect of the original function.
Definition of an inverse function: If a function f takes x and produces $f(x)$, then the **inverse function** f^{-1} reverses this process: it takes $f(x)$ and returns x.That is, applying f and then f^{-1} brings you back to where you started, and vice versa.

Definition Inverse Function

A function f^{-1} is called the **inverse function** of f if:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

When does an inverse exist?
A function f has an inverse function f^{-1} **only if**, for every value of $f(x)$, there is exactly *one* value of x that produces it (i.e., f is **one-to-one**).

Method Finding the Inverse Function

To find the inverse of $f(x)$:

1. Set $y = f(x)$.
2. Solve the equation for x in terms of y.
3. Replace y with x; the result is $f^{-1}(x)$.

Ex: Find the inverse of $f(x) = 4x - 8$.

Answer: Set $y = 4x - 8$.

$$\begin{aligned} y &= 4x - 8 \\ y + 8 &= 4x \\ x &= \frac{y+8}{4} \end{aligned}$$

So, the inverse function is $f^{-1}(x) = \frac{x+8}{4}$.

Proposition Symmetry of the Graphs of a Function and Its Inverse

The graph of a function f and its inverse f^{-1} are symmetric with respect to the line $y = x$ (the first bisector).