FUNCTIONS AND GRAPHS OF POLYNOMIALS

A GENERAL PRINCIPLES OF POLYNOMIAL GRAPHS

A.1 DETERMINING END BEHAVIOUR

Ex 1: For the polynomial $P(x) = -3x^5 + 4x^2 - 8x + 1$, determine the following limits:

1. $\lim_{x\to\infty} P(x) = \boxed{-\infty}$
2. $\lim_{x\to-\infty} P(x) = \boxed{+\infty}$

Answer: To determine the end behaviour, we factor out the leading term, $-3x^5$, from the polynomial:

$$\begin{aligned}P(x) &= -3x^5 + 4x^2 - 8x + 1\\ &= -3x^5\left(1 - \frac{4x^2}{3x^5} + \frac{8x}{3x^5} - \frac{1}{3x^5}\right)\\ &= -3x^5\left(1 - \frac{4}{3x^3} + \frac{8}{3x^4} - \frac{1}{3x^5}\right)\end{aligned}$$

As $x \to \pm\infty$, all the fractional terms inside the parentheses approach 0. This means the entire expression in the parentheses approaches 1.
Therefore, for very large values of $|x|$, the behaviour of $P(x)$ is determined by the behaviour of its leading term, $-3x^5$.
The degree is $n = 5$ (odd) and the leading coefficient is $a_n = -3$ (negative).

1. As $x \to \infty$, an odd power of a large positive number is positive, but it is multiplied by a negative coefficient. Thus, $P(x) \to -\infty$.
2. As $x \to -\infty$, an odd power of a large negative number is negative. Multiplying by a negative coefficient (-3) makes the result positive. Thus, $P(x) \to \infty$.

The graph runs from top-left to bottom-right.

Ex 2: For the polynomial $P(x) = 2x^4 - 5x^3 + x - 10$, determine the following limits:

1. $\lim_{x\to\infty} P(x) = \boxed{+\infty}$
2. $\lim_{x\to-\infty} P(x) = \boxed{+\infty}$

Answer: To determine the end behaviour, we factor out the leading term, $2x^4$:

$$\begin{aligned}P(x) &= 2x^4 - 5x^3 + x - 10\\ &= 2x^4\left(1 - \frac{5x^3}{2x^4} + \frac{x}{2x^4} - \frac{10}{2x^4}\right)\\ &= 2x^4\left(1 - \frac{5}{2x} + \frac{1}{2x^3} - \frac{5}{x^4}\right)\end{aligned}$$

As $x \to \pm\infty$, all the fractional terms inside the parentheses approach 0. This means the entire expression in the parentheses approaches 1.
Therefore, for very large values of $|x|$, the behaviour of $P(x)$ is determined by the behaviour of its leading term, $2x^4$.
The degree is $n = 4$ (even) and the leading coefficient is $a_n = 2$ (positive).

1. As $x \to \infty$, an even power of a large positive number is positive. Multiplying by a positive coefficient keeps it positive. Thus, $P(x) \to \infty$.
2. As $x \to -\infty$, an even power of a large negative number is positive. Multiplying by a positive coefficient keeps it positive. Thus, $P(x) \to \infty$.

The graph opens upwards, running from top-left to top-right.

Ex 3: For the polynomial $P(x) = x^7 + 100x^6 - 500x^2$, determine the following limits:

1. $\lim_{x\to\infty} P(x) = \boxed{+\infty}$
2. $\lim_{x\to-\infty} P(x) = \boxed{-\infty}$

Answer: To determine the end behaviour, we factor out the leading term, x^7:

$$\begin{aligned}P(x) &= x^7 + 100x^6 - 500x^2\\ &= x^7\left(1 + \frac{100x^6}{x^7} - \frac{500x^2}{x^7}\right)\\ &= x^7\left(1 + \frac{100}{x} - \frac{500}{x^5}\right)\end{aligned}$$

As $x \to \pm\infty$, all the fractional terms inside the parentheses approach 0. This means the entire expression in the parentheses approaches 1.
Therefore, for very large values of $|x|$, the behaviour of $P(x)$ is determined by the behaviour of its leading term, x^7.
The degree is $n = 7$ (odd) and the leading coefficient is $a_n = 1$ (positive).

1. As $x \to \infty$, an odd power of a large positive number is positive. Thus, $P(x) \to \infty$.
2. As $x \to -\infty$, an odd power of a large negative number is negative. Thus, $P(x) \to -\infty$.

The graph runs from bottom-left to top-right.

Ex 4: For the polynomial $P(x) = 50 + x - 2x^6$, determine the following limits:

1. $\lim_{x\to\infty} P(x) = \boxed{-\infty}$
2. $\lim_{x\to-\infty} P(x) = \boxed{-\infty}$

Answer: First, we write the polynomial in standard form: $P(x) = -2x^6 + x + 50$. To determine the end behaviour, we factor out the leading term, $-2x^6$:

$$\begin{aligned}P(x) &= -2x^6 + x + 50\\ &= -2x^6\left(1 - \frac{x}{2x^6} - \frac{50}{2x^6}\right)\\ &= -2x^6\left(1 - \frac{1}{2x^5} - \frac{25}{x^6}\right)\end{aligned}$$

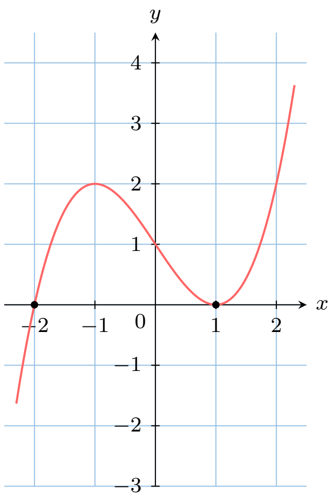
As $x \to \pm\infty$, all the fractional terms inside the parentheses approach 0. This means the entire expression in the parentheses approaches 1.
Therefore, for very large values of $|x|$, the behaviour of $P(x)$ is determined by the behaviour of its leading term, $-2x^6$.
The degree is $n = 6$ (even) and the leading coefficient is $a_n = -2$ (negative).

1. As $x \to \infty$, an even power of a large positive number is positive. Multiplying by a negative coefficient makes it negative. Thus, $P(x) \to -\infty$.
2. As $x \to -\infty$, an even power of a large negative number is positive. Multiplying by a negative coefficient makes it negative. Thus, $P(x) \to -\infty$.

The graph opens downwards, running from bottom-left to bottom-right.

A.2 INTERPRETING GRAPHS AT THE ROOTS

MCQ 5: The graph of a polynomial $P(x)$ is shown below. Which of the following is the most likely factorisation of $P(x)$?



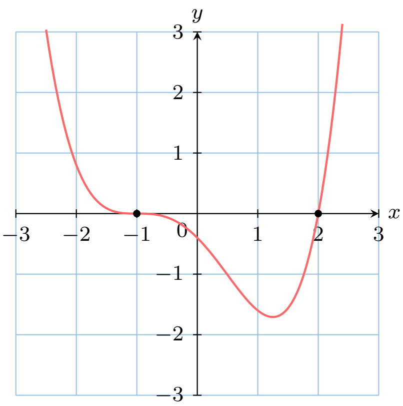
- ☐ $(x+2)(x-1)$
- ☐ $(x+2)^2(x-1)$
- ☒ $(x+2)(x-1)^2$
- ☐ $(x+2)^3(x-1)$

Answer: The graph has two x-intercepts (roots) at $x = -2$ and $x = 1$.

- At $x = -2$, the graph **cuts** the axis, which implies the factor $(x+2)$ has an odd multiplicity (likely 1 or 3).
- At $x = 1$, the graph **touches** the axis, which implies the factor $(x-1)$ has an even multiplicity (likely 2).

Combining these observations, the most likely factorisation is $P(x) = a(x+2)(x-1)^2$ for some constant a.

MCQ 6: The graph of a polynomial $P(x)$ is shown below. Which of the following is the most likely factorisation of $P(x)$?



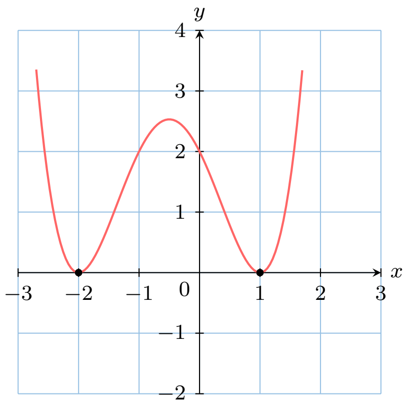
- ☐ $(x+1)(x-2)$
- ☐ $(x+1)^2(x-2)$
- ☐ $(x+1)(x-2)^3$
- ☒ $(x+1)^3(x-2)$

Answer: The graph has two x-intercepts at $x = -1$ and $x = 2$.

- At $x = -1$, the graph has a **point of inflection with a horizontal tangent**, which implies the factor $(x+1)$ has a multiplicity of 3.
- At $x = 2$, the graph **cuts** the axis, which implies the factor $(x-2)$ has a multiplicity of 1.

Combining these observations, the most likely factorisation is $P(x) = a(x+1)^3(x-2)$ for some constant a.

MCQ 7: The graph of a polynomial $P(x)$ is shown below. Which of the following is the most likely factorisation of $P(x)$?



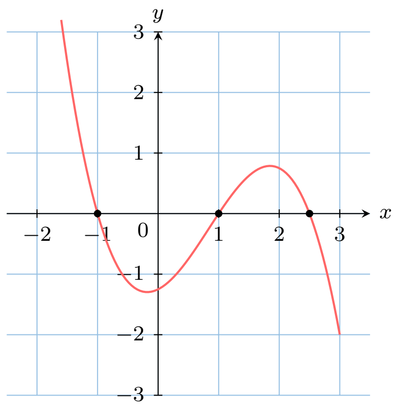
- ☐ $(x+2)(x-1)$
- ☐ $(x-2)(x+1)^2$
- ☒ $(x+2)^2(x-1)^2$
- ☐ $(x+2)^3(x-1)$

Answer: The graph has two x-intercepts at $x = -2$ and $x = 1$.

- At $x = -2$, the graph **touches** the axis, which implies the factor $(x+2)$ has an even multiplicity (likely 2).
- At $x = 1$, the graph also **touches** the axis, which implies the factor $(x-1)$ has an even multiplicity (likely 2).

Combining these observations, the most likely factorisation is $P(x) = a(x+2)^2(x-1)^2$ for some constant a.

MCQ 8: The graph of a polynomial $P(x)$ is shown below. Which of the following is the most likely factorisation of $P(x)$?



☐ $(x+1)(x-1)(x-2.5)$

☒ $-(x+1)(x-1)(x-2.5)$

☐ $-(x-1)(x+1)^2$

☐ $(x+1)(x-1)(x-2.5)^2$

Answer: The graph has three x-intercepts at $x=-1, x=1$, and $x=2.5$.

- At each root, the graph **cuts** the axis, which implies that each corresponding factor, $(x+1)$, $(x-1)$, and $(x-2.5)$, has a multiplicity of 1.
- We also observe the end behaviour. As $x \to \infty$, $y \to -\infty$. This indicates a negative leading coefficient.

Combining these observations, the most likely factorisation is $P(x) = a(x+1)(x-1)(x-2.5)$ where $a < 0$. Option B matches this form.

A.3 INTERPRETING GRAPHS AT THE ROOTS

Ex 9: A polynomial function is given by $P(x) = (x-1)^3(x-2)$. Describe the behaviour of the graph of $y = P(x)$ at its x-intercepts.

Answer: The roots of the polynomial are at $x=1$ and $x=2$.

- The root $x=1$ comes from the factor $(x-1)^3$, which has a multiplicity of 3. Therefore, the graph has a **point of horizontal inflection** on the x-axis at $x=1$.
- The root $x=2$ comes from the factor $(x-2)^1$, which has a multiplicity of 1. Therefore, the graph **cuts** the x-axis at $x=2$.

Ex 10: A polynomial function is given by $P(x) = (x+3)^2(x-4)$. Describe the behaviour of the graph of $y = P(x)$ at its x-intercepts.

Answer: The roots of the polynomial are at $x=-3$ and $x=4$.

- The root $x=-3$ comes from the factor $(x+3)^2$, which has a multiplicity of 2. Therefore, the graph **touches** the x-axis at $x=-3$.
- The root $x=4$ comes from the factor $(x-4)^1$, which has a multiplicity of 1. Therefore, the graph **cuts** the x-axis at $x=4$.

Ex 11: A polynomial function is given by $P(x) = x(x-5)^2(x+1)^3$. Describe the behaviour of the graph of $y = P(x)$ at its x-intercepts.

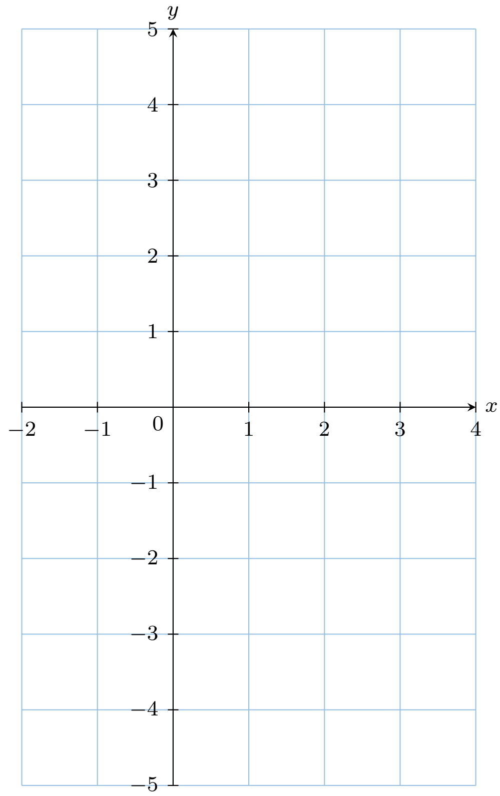
Answer: The roots of the polynomial are at $x=0$, $x=5$, and $x=-1$.

- The root $x=0$ comes from the factor x^1, which has a multiplicity of 1. Therefore, the graph **cuts** the x-axis at the origin.
- The root $x=5$ comes from the factor $(x-5)^2$, which has a multiplicity of 2. Therefore, the graph **touches** the x-axis at $x=5$.
- The root $x=-1$ comes from the factor $(x+1)^3$, which has a multiplicity of 3. Therefore, the graph has a **point of horizontal inflection** on the x-axis at $x=-1$.

B GRAPHING CUBIC FUNCTIONS

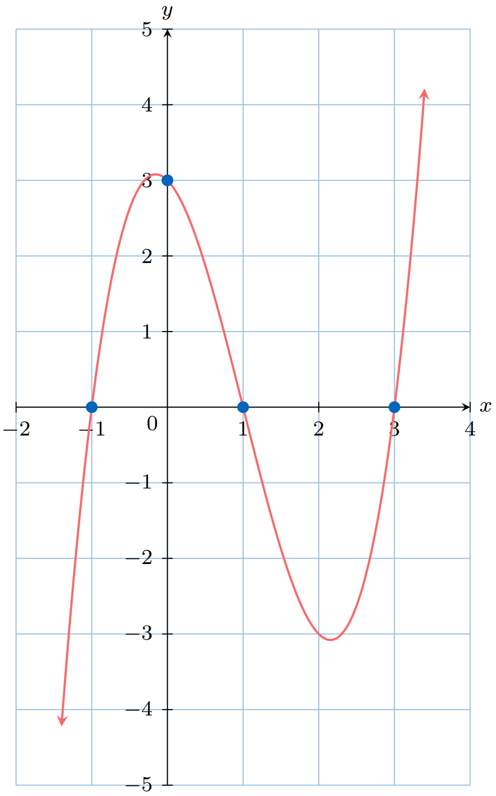
B.1 SKETCHING CUBIC FUNCTIONS

Ex 12: Use the axes intercepts to sketch the graph of $y = (x+1)(x-1)(x-3)$.

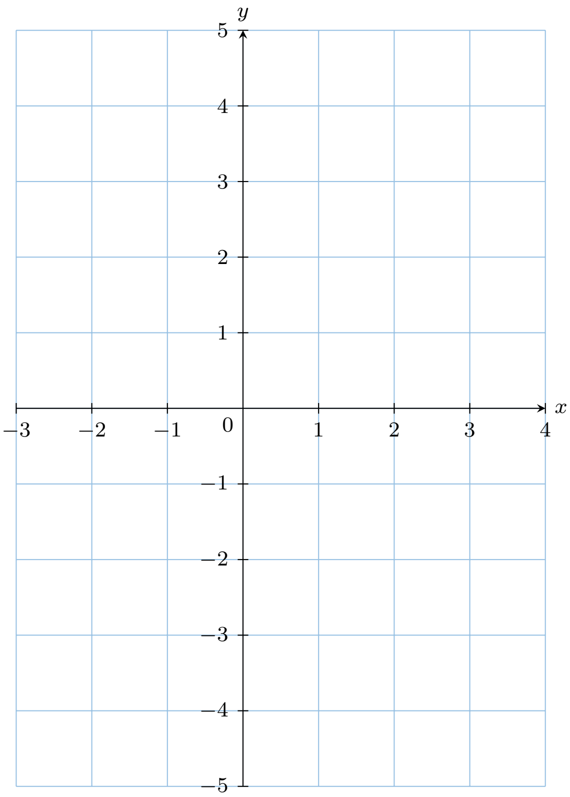


Answer:

1. **Roots**: The polynomial is fully factored with three distinct real roots. The graph **cuts** the x-axis at $x=-1, x=1$, and $x=3$.
2. **y-intercept**: When $x=0$, $y=(1)(-1)(-3)=3$. The y-intercept is $(0,3)$.
3. **End Behaviour**: The leading term is $(x)(x)(x)=x^3$. The degree is odd and the leading coefficient is positive. Thus, as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$.
4. **Sketch**:

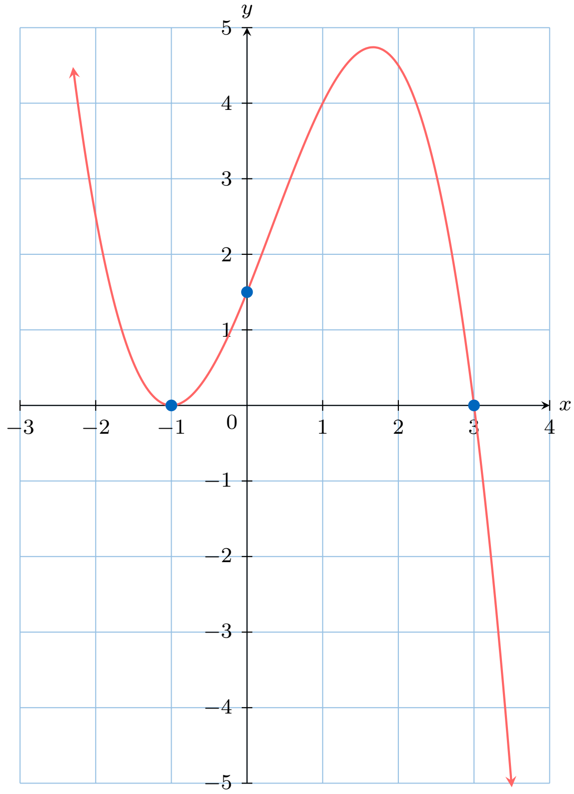


Ex 13: Use the axes intercepts to sketch the graph of $y = -\frac{1}{2}(x+1)^2(x-3)$.

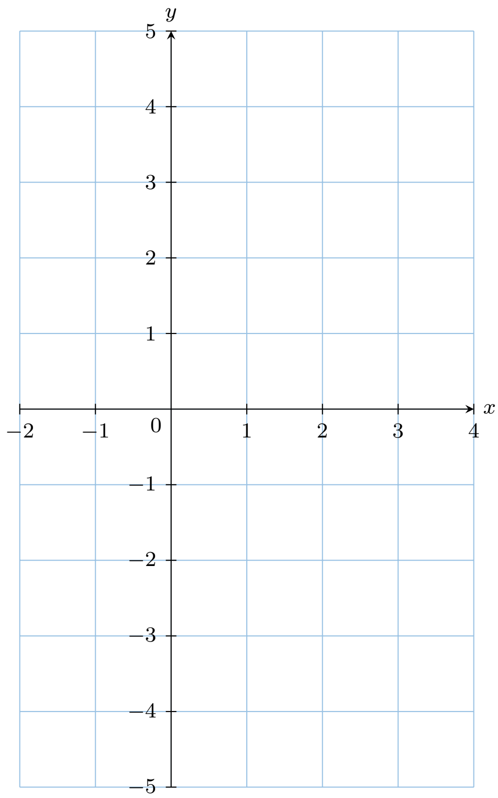


Answer:

1. **Roots**: The polynomial is factored. There is a repeated root at $x = -1$ (from the $(x+1)^2$ term), so the graph **touches** the x-axis at this point. There is a single root at $x = 3$, so the graph **cuts** the x-axis at this point.
2. **y-intercept**: When $x = 0$, $y = -\frac{1}{2}(0+1)^2(0-3) = -\frac{1}{2}(1)(-3) = \frac{3}{2}$. The y-intercept is $(0, 1.5)$.
3. **End Behaviour**: The leading term is $-\frac{1}{2}(x^2)(x) = -\frac{1}{2}x^3$. The degree is odd and the leading coefficient is negative. Thus, as $x \to \infty, y \to -\infty$ and as $x \to -\infty, y \to \infty$.
4. **Sketch**:



Ex 14: Use the axes intercepts to sketch the graph of $y = \frac{1}{2}(x-1)^3$.



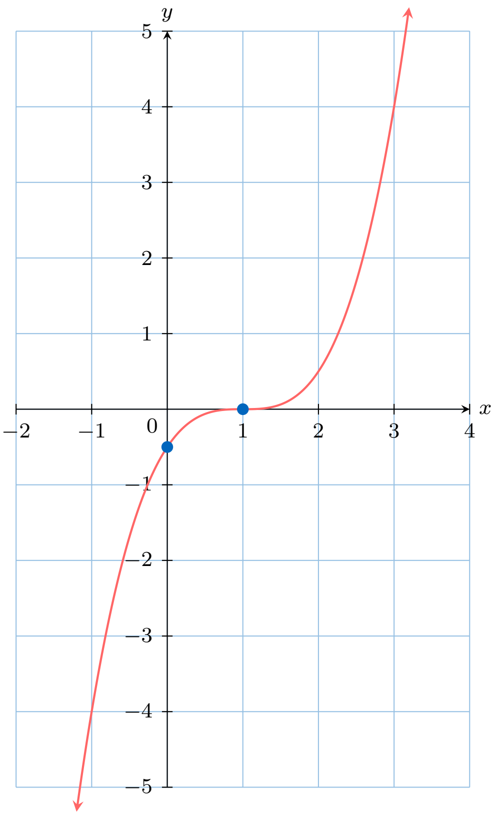
Answer:

1. **Roots**: The polynomial has a root of multiplicity 3 at $x = 1$. This means the graph **cuts** the x-axis at $x = 1$ with a **stationary point of inflection**.

2. **y-intercept**: When $x = 0$, $y = \frac{1}{2}(0-1)^3 = \frac{1}{2}(-1) = -\frac{1}{2}$. The y-intercept is $(0, -0.5)$.

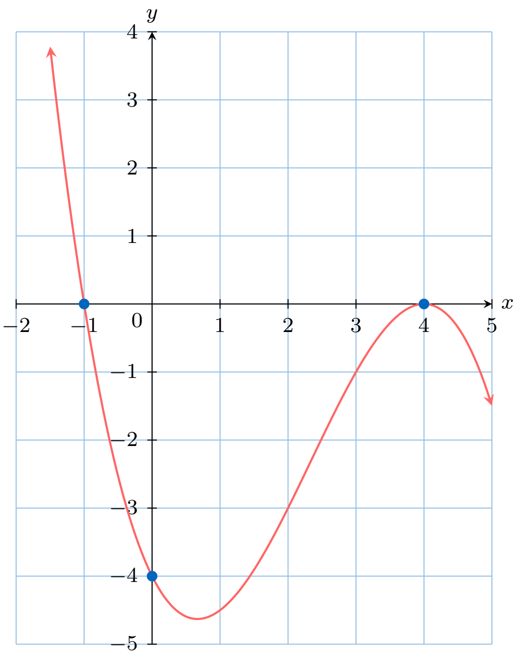
3. **End Behaviour**: The leading term is $\frac{1}{2}x^3$. The degree is odd and the leading coefficient is positive. Thus, as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$.

4. **Sketch**:



B.2 FINDING THE FUNCTION FROM A GRAPH

Ex 15: Find the equation of the cubic function shown in the graph below.



$P(x) =$ $\boxed{-1/4(x+1)(x-4)^2}$

Answer:

1. **Identify the roots from the graph**:
   - The graph **cuts** the x-axis at $x = -1$. This corresponds to a single root, giving a factor of $(x - (-1)) = (x + 1)$.
   - The graph **touches** the x-axis at $x = 4$. This indicates a repeated root of even multiplicity. For a cubic function, this must be a root of multiplicity 2, giving a factor of $(x-4)^2$.

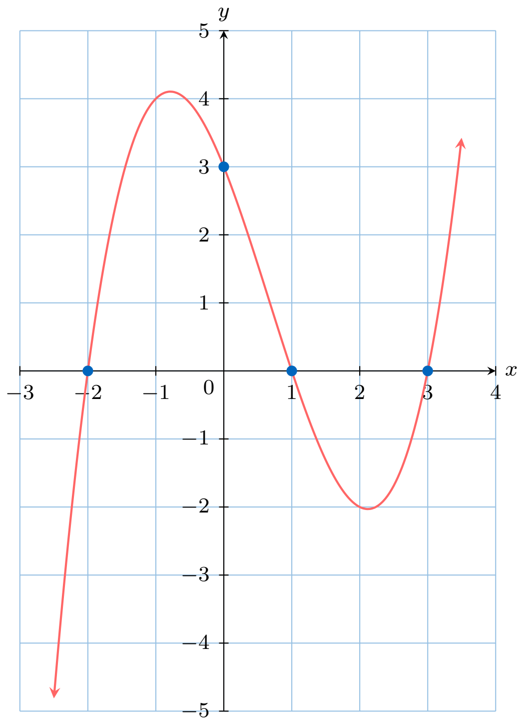
2. **Write the general equation of the function**: Based on the roots, the equation of the cubic function can be written in the form: $y = a(x+1)(x-4)^2$, where a is a constant.

3. **Use the y-intercept to find the value of *a***: The graph passes through the point $(0, -4)$. Substitute $x = 0$ and $y = -4$ into the equation:

$$\begin{aligned}-4 &= a(0+1)(0-4)^2\\ -4 &= a(1)(-4)^2\\ -4 &= a(16)\\ a &= \frac{-4}{16}\\ a &= -\frac{1}{4}\end{aligned}$$

4. **Write the final equation**: Substituting the value of a back into the general equation, we get: $y = -\frac{1}{4}(x+1)(x-4)^2$

Ex 16: Find the equation of the cubic function shown in the graph below.



$P(x) =$ $\boxed{1/2(x+2)(x-1)(x-3)}$

Answer:

1. **Identify the roots from the graph**:
   - The graph **cuts** the x-axis at $x = -2$, $x = 1$, and $x = 3$. These are all single roots.
   - The corresponding factors are $(x+2)$, $(x-1)$, and $(x-3)$.

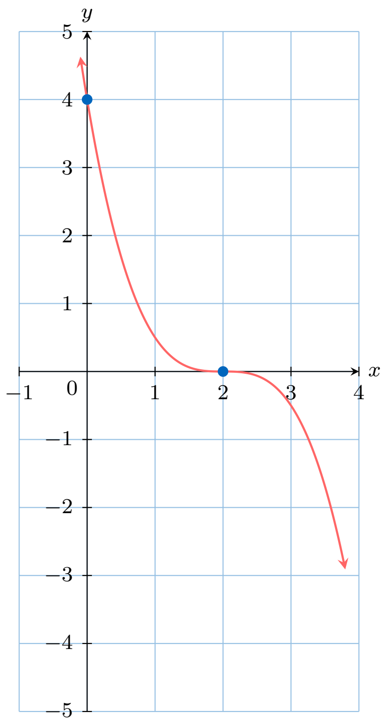
2. **Write the general equation of the function**: Based on the roots, the equation of the cubic function can be written in the form: $y = a(x+2)(x-1)(x-3)$, where a is a constant.

3. **Use the y-intercept to find the value of a**: The graph passes through the point $(0, 3)$. Substitute $x = 0$ and $y = 3$ into the equation:

$$3 = a(0+2)(0-1)(0-3)$$
$$3 = a(2)(-1)(-3)$$
$$3 = a(6)$$
$$a = \frac{3}{6}$$
$$a = \frac{1}{2}$$

4. **Write the final equation**: Substituting the value of a back into the general equation, we get: $y = \frac{1}{2}(x+2)(x-1)(x-3)$

Ex 17: Find the equation of the cubic function shown in the graph below.



$P(x) = \boxed{-1/2(x-2)^3}$

Answer:

1. **Identify the roots from the graph**:
   - The graph **cuts** the x-axis at $x = 2$ with a **stationary point of inflection**. This indicates a root of multiplicity 3.
   - The corresponding factor is $(x-2)^3$.

2. **Write the general equation of the function**: Based on the root, the equation of the cubic function can be written in the form: $y = a(x-2)^3$, where a is a constant.

3. **Use the y-intercept to find the value of a**: The graph passes through the point $(0, 4)$. Substitute $x = 0$ and $y = 4$ into the equation:

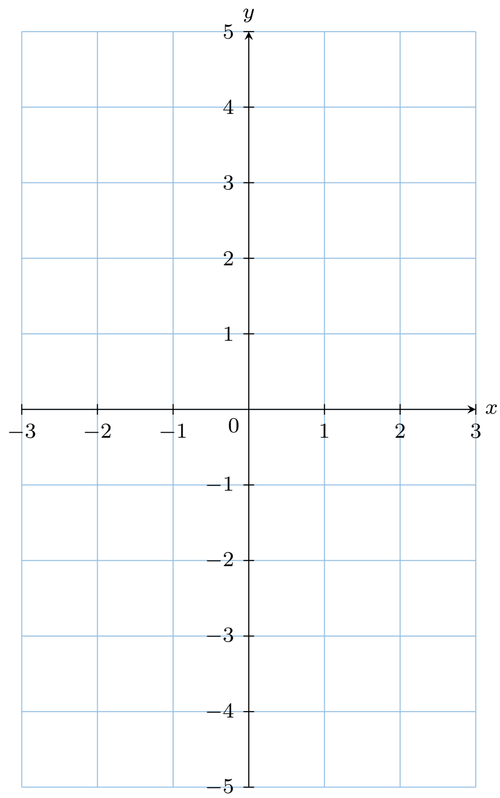
$$4 = a(0-2)^3$$
$$4 = a(-8)$$
$$a = \frac{4}{-8}$$
$$a = -\frac{1}{2}$$

4. **Write the final equation**: Substituting the value of a back into the general equation, we get: $y = -\frac{1}{2}(x-2)^3$

C GRAPHING QUARTIC FUNCTIONS

C.1 SKETCHING QUARTIC FUNCTIONS

Ex 18: Use the axes intercepts to sketch the graph of $y = (x+2)(x+1)(x-1)(x-2)$.



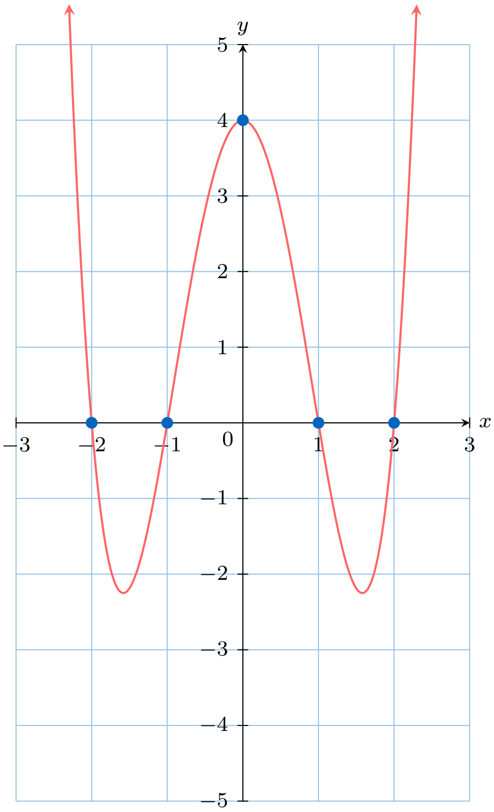
Answer:

1. **Roots**: The polynomial is fully factored with four distinct real roots. The graph **cuts** the x-axis at $x = -2, x = -1, x = 1$, and $x = 2$.

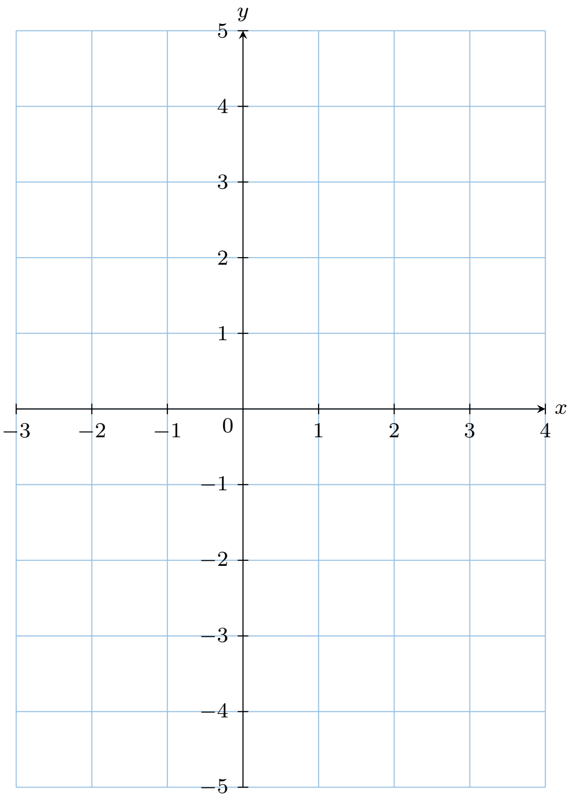
2. **y-intercept**: When $x = 0$, $y = (2)(1)(-1)(-2) = 4$. The y-intercept is $(0, 4)$.

3. **End Behaviour**: The leading term is $(x)(x)(x)(x) = x^4$. The degree is even and the leading coefficient is positive. Thus, as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to \infty$.

4. **Sketch**:



Ex 19: Use the axes intercepts to sketch the graph of $y = -\frac{1}{4}(x+2)^2(x-1)(x-3)$.



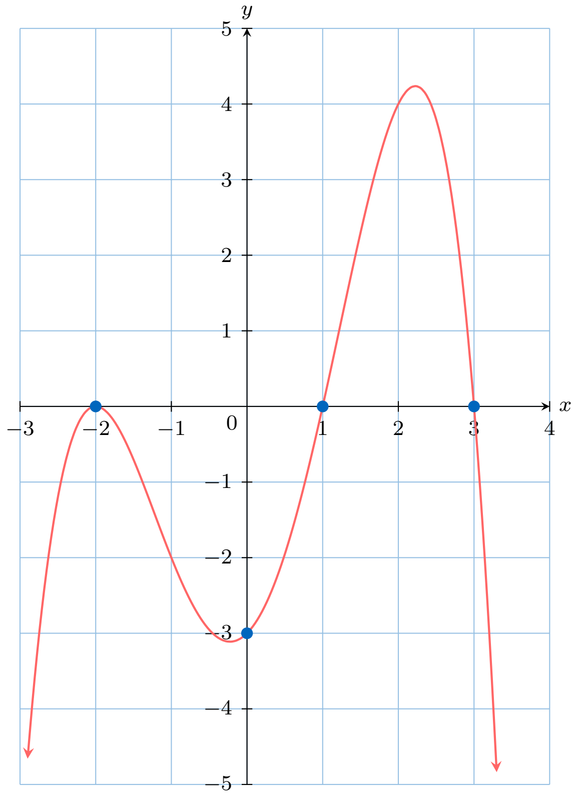
Answer:

1. **Roots**: The polynomial is factored. There is a repeated root at $x = -2$ (multiplicity 2), so the graph **touches** the x-axis at this point. There are single roots at $x = 1$ and $x = 3$, where the graph **cuts** the x-axis.

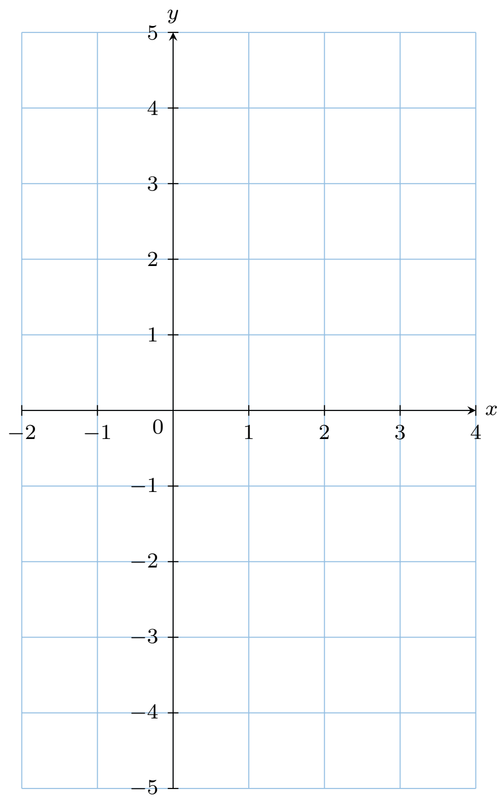
2. **y-intercept**: When $x = 0$, $y = -\frac{1}{4}(0+2)^2(0-1)(0-3) = -\frac{1}{4}(4)(-1)(-3) = -3$. The y-intercept is $(0, -3)$.

3. **End Behaviour**: The leading term is $-\frac{1}{4}(x^2)(x)(x) = -\frac{1}{4}x^4$. The degree is even and the leading coefficient is negative. Thus, as $x \to \infty, y \to -\infty$ and as $x \to -\infty, y \to -\infty$.

4. **Sketch**:

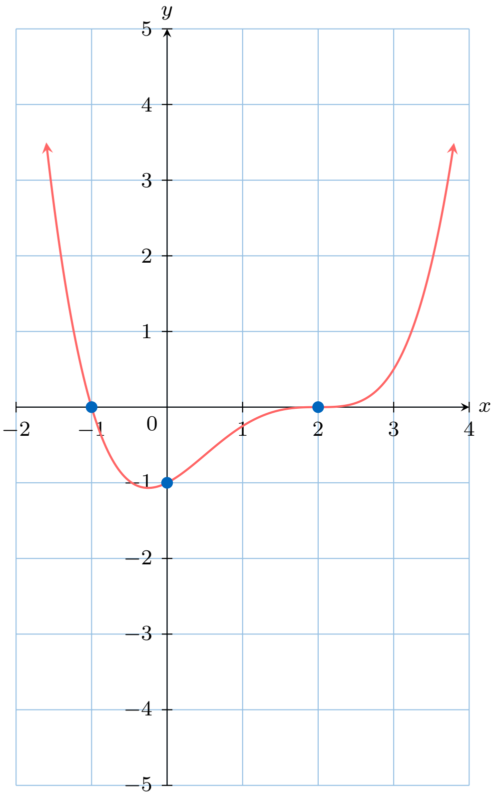


Ex 20: Use the axes intercepts to sketch the graph of $y = \frac{1}{8}(x-2)^3(x+1)$.



Answer:

1. **Roots**: The polynomial has a root of multiplicity 3 at $x = 2$. This means the graph **cuts** the x-axis at $x = 2$ with a **stationary point of inflection**. There is a single root at $x = -1$, where the graph **cuts** the x-axis.
2. **y-intercept**: When $x = 0$, $y = \frac{1}{8}(0-2)^3(0+1) = \frac{1}{8}(-8)(1) = -1$. The y-intercept is $(0, -1)$.
3. **End Behaviour**: The leading term is $\frac{1}{8}(x^3)(x) = \frac{1}{8}x^4$. The degree is even and the leading coefficient is positive. Thus, as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to \infty$.
4. **Sketch**:



D SOLVING POLYNOMIAL INEQUALITIES

D.1 SOLVING POLYNOMIAL INEQUALITIES

MCQ 21: Which of the following is the solution to the inequality $(x-4)^2(x+1) \leq 0$?

☐ $x \leq -1$

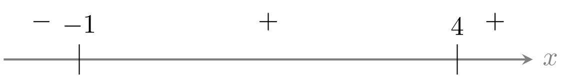
☐ $x \leq -1$ or $x \geq 4$

☒ $x \leq -1$ or $x = 4$

☐ $-1 \leq x \leq 4$

Answer: Let $P(x) = (x-4)^2(x+1)$. The roots are at $x = 4$ (multiplicity 2) and $x = -1$ (multiplicity 1).

- **Method 1: Sign Diagram**
  We mark the roots on a number line. Because of the squared term $(x-4)^2$, the sign of $P(x)$ does not change at $x = 4$. The sign is determined by the linear factor $(x+1)$.



- **Method 2: Table of Signs**
  We analyze the sign of each factor. Note that $(x-4)^2$ is always non-negative.

x	$-\infty$		-1		4		$+\infty$
$(x-4)^2$		$+$		$+$	0	$+$	
$x+1$		$-$	0	$+$		$+$	
$P(x)$		$-$	0	$+$	0	$+$	

Conclusion:
We are looking for where $P(x) \leq 0$.
The sign diagrams show that $P(x)$ is negative for $x < -1$.
The polynomial is equal to zero at its roots, $x = -1$ and $x = 4$.
Combining these conditions ($P(x) < 0$ or $P(x) = 0$), the solution is $\boldsymbol{x \leq -1}$ **or** $\boldsymbol{x = 4}$.

MCQ 22: Which of the following is the solution to the inequality $(x+3)(x-1)^3 \geq 0$?

☐ $-3 \leq x \leq 1$

☐ $x \leq -3$

☐ $x \geq 1$

☒ $x \leq -3$ or $x \geq 1$

Answer: Let $P(x) = (x+3)(x-1)^3$. The roots are at $x = -3$ (multiplicity 1) and $x = 1$ (multiplicity 3).

- **Method 1: Sign Diagram**
  We mark the roots on a number line. Since both factors have odd multiplicity, the sign of $P(x)$ will change at both roots. We can test a value, for instance $x = 2$: $(2+3)(2-1)^3 = (5)(1)^3 > 0$.

- **Method 2: Table of Signs**
  We analyze the sign of each factor. The sign of $(x-1)^3$ is the same as the sign of $(x-1)$.

x	$-\infty$		-3		1		$+\infty$
$x+3$		$-$	0	$+$		$+$	
$(x-1)^3$		$-$		$-$	0	$+$	
$P(x)$		$+$	0	$-$	0	$+$	

Conclusion:
We are looking for where $P(x) \geq 0$.
The sign diagrams show that $P(x)$ is positive for $x < -3$ and for $x > 1$.
The polynomial is equal to zero at its roots, $x = -3$ and $x = 1$.
Combining these conditions ($P(x) > 0$ or $P(x) = 0$), the solution is $\boldsymbol{x \leq -3}$ **or** $\boldsymbol{x \geq 1}$.

MCQ 23: Which of the following is the solution to the inequality $(x+3)(x-1)^3 \geq 0$?

☐ $-3 \le x \le 1$

☐ $x \le -3$

☐ $x \ge 1$

☒ $x \le -3$ or $x \ge 1$

Answer: Let $P(x) = (x+3)(x-1)^3$. The roots are at $x = -3$ (multiplicity 1) and $x = 1$ (multiplicity 3).

- **Method 1: Sign Diagram**
  We mark the roots on a number line. Since both factors have odd multiplicity, the sign of $P(x)$ will change at both roots. We can test a value, for instance $x = 2$: $(2+3)(2-1)^3 = (5)(1)^3 > 0$.

- **Method 2: Table of Signs**
  We analyze the sign of each factor. The sign of $(x-1)^3$ is the same as the sign of $(x-1)$.

x	$-\infty$		-3		1		$+\infty$
$x+3$		$-$	0	$+$		$+$	
$(x-1)^3$		$-$		$-$	0	$+$	
$P(x)$		$+$	0	$-$	0	$+$	

Conclusion:
We are looking for where $P(x) \ge 0$.
The sign diagrams show that $P(x)$ is positive for $x < -3$ and for $x > 1$.
The polynomial is equal to zero at its roots, $x = -3$ and $x = 1$.
Combining these conditions ($P(x) > 0$ or $P(x) = 0$), the solution is $\boldsymbol{x \le -3}$ or $\boldsymbol{x \ge 1}$.

MCQ 24: Which of the following is the solution to the inequality $x^3 + 2x^2 < 3x$?

☐ $x < -3$ or $x > 1$

☒ $x < -3$ or $0 < x < 1$

☐ $-3 < x < 0$

☐ $-3 < x < 0$ or $x > 1$

Answer: First, we rearrange the inequality to have zero on one side.

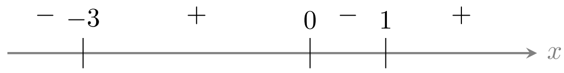
$$x^3 + 2x^2 - 3x < 0$$

Let $P(x) = x^3 + 2x^2 - 3x$. The roots are found by factoring:

$$P(x) = x(x^2 + 2x - 3) = x(x+3)(x-1)$$

The roots are at $x = -3$, $x = 0$, and $x = 1$.

- **Method 1: Sign Diagram**
  We mark the roots on a number line. All roots have multiplicity 1, so the sign will change at each root. We can test a value, for instance $x = 2$: $2(5)(1) > 0$.



- **Method 2: Table of Signs**
  We analyze the sign of each factor.

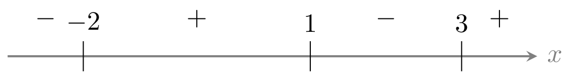
x	$-\infty$		-3		0		1		$+\infty$
$x+3$		$-$	0	$+$		$+$		$+$	
x		$-$		$-$	0	$+$		$+$	
$x-1$		$-$		$-$		$-$	0	$+$	
$P(x)$		$-$	0	$+$	0	$-$	0	$+$	

Conclusion:
We are looking for where $P(x) < 0$.
The sign diagrams show that $P(x)$ is negative for $x < -3$ and for $0 < x < 1$.
Since the inequality is strict ($<$), the endpoints are not included in the solution.
The solution is $\boldsymbol{x < -3}$ or $\boldsymbol{0 < x < 1}$.

Ex 25: Find the set of values for which $(x+2)(x-1)(x-3) < 0$.

Answer: Let $P(x) = (x+2)(x-1)(x-3)$.

1. **Find roots**: The polynomial is already factorised. The roots are $x = -2, x = 1, x = 3$.
2. - **Method 1: Draw Sign Diagram**: We mark the roots on a number line and test a value in each interval. For example, test $x = 4$: $(+)(+)(+) > 0$.



   - **Method 2: Draw Table of Signs:** We analyse the sign of each individual factor across the intervals before determining the sign of the final product.

x	$-\infty$		-2		1		3		$+\infty$
$x+2$		$-$	0	$+$		$+$		$+$	
$x-1$		$-$		$-$	0	$+$		$+$	
$x-3$		$-$		$-$		$-$	0	$+$	
$P(x)$		$-$	0	$+$	0	$-$	0	$+$	

3. **State Solution**: We are looking for where $P(x) < 0$. The sign diagram shows this is true for $x < -2$ and for $1 < x < 3$. The solution is $\boldsymbol{(-\infty, -2) \cup (1, 3)}$.

Ex 26: Find the set of values for which $x^2 - 3x + 2 > 0$.

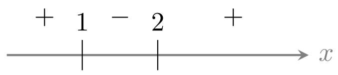
Answer: Let $P(x) = x^2 - 3x + 2$.

1. **Find roots**: First, we factorise the polynomial.

$$P(x) = (x-1)(x-2)$$

The roots are $x = 1$ and $x = 2$.

2. - **Method 1: Draw Sign Diagram**: We mark the roots on a number line and test a value in each interval. For example, test $x = 3$: $(3-1)(3-2) = (2)(1) > 0$.



   - **Method 2: Draw Table of Signs**: We analyse the sign of each factor.

x	$-\infty$		1		2		$+\infty$
$x-1$		$-$	0	$+$		$+$	
$x-2$		$-$		$-$	0	$+$	
$P(x)$		$+$	0	$-$	0	$+$	

3. **State Solution**: We are looking for where $P(x) > 0$. This occurs for $x < 1$ and for $x > 2$.
The solution is $\mathbf{(-\infty, 1) \cup (2, \infty)}$.

D.2 SOLVING POLYNOMIAL AND RATIONAL INEQUALITIES

Ex 27: Consider the inequality $\frac{x-4}{2x+1} \geq 1$.

1. Rewrite the inequality in the form $f(x) \geq 0$, where $f(x)$ is a single rational expression.
2. Find the critical values for the inequality.
3. Hence, solve the inequality $\frac{x-4}{2x+1} \geq 1$.

Answer:

1. To rewrite the inequality, we first subtract 1 from both sides and then find a common denominator:

$$\frac{x-4}{2x+1} - 1 \geq 0$$
$$\frac{x-4-(2x+1)}{2x+1} \geq 0$$
$$\frac{x-4-2x-1}{2x+1} \geq 0$$
$$\frac{-x-5}{2x+1} \geq 0$$

So, $f(x) = \frac{-x-5}{2x+1}$.

2. The critical values are the values of x for which the numerator or the denominator is zero.
   - Numerator: $-x-5 = 0 \implies x = -5$.
   - Denominator: $2x+1 = 0 \implies x = -\frac{1}{2}$.

   The critical values are $x = -5$ and $x = -1/2$.

3. We use a sign table to determine the intervals where $f(x) \geq 0$.

x	$-\infty$		-5		$-1/2$		$+\infty$
$-x-5$		$+$	0	$-$		$-$	
$2x+1$		$-$		$-$	‖	$+$	
$f(x)$		$-$	0	$+$	‖	$-$	

We are looking for intervals where $f(x) \geq 0$. From the table, this is the interval between -5 and -1/2.
We must also consider the equality. $f(x) = 0$ when the numerator is zero, which is at $x = -5$. So, $x = -5$ is included in the solution.
The denominator cannot be zero, so $x = -1/2$ is excluded from the solution (indicated by the double bar in the table).
Therefore, the solution is $[-5, -1/2)$.

Ex 28: Solve the inequality $x \leq \frac{3}{x-2}$.

1. Rewrite the inequality in the form $f(x) \leq 0$, where $f(x)$ is a single rational expression.
2. Find the critical values for the inequality.
3. Hence, solve the inequality $x \leq \frac{3}{x-2}$.

Answer:

1. To rewrite the inequality, we move all terms to one side and combine them into a single fraction:

$$x - \frac{3}{x-2} \leq 0$$
$$\frac{x(x-2)-3}{x-2} \leq 0$$
$$\frac{x^2-2x-3}{x-2} \leq 0$$
$$\frac{(x-3)(x+1)}{x-2} \leq 0$$

So, $f(x) = \frac{(x-3)(x+1)}{x-2}$.

2. The critical values are the values of x for which the numerator or the denominator is zero.
   - Numerator: $(x-3)(x+1) = 0 \implies x = 3$ or $x = -1$.
   - Denominator: $x-2 = 0 \implies x = 2$.

   The critical values are $x = -1$, $x = 2$, and $x = 3$.

3. We use a sign table to determine the intervals where $f(x) \leq 0$.

x	$-\infty$		-1		2		3		$+\infty$
$x+1$		$-$	0	$+$		$+$		$+$	
$x-2$		$-$		$-$	‖	$+$		$+$	
$x-3$		$-$		$-$		$-$	0	$+$	
$f(x)$		$-$	0	$+$	‖	$-$	0	$+$	

We are looking for intervals where $f(x) \leq 0$. From the table, these are the intervals $(-\infty, -1)$ and $(2, 3)$.
We must also consider the equality. $f(x) = 0$ when the numerator is zero, which occurs at $x = -1$ and $x = 3$. So, these values are included in the solution.
The denominator cannot be zero, so $x = 2$ is excluded from the solution.
Therefore, the solution is $(-\infty, -1] \cup (2, 3]$.

Ex 29: Solve the inequality $\frac{2}{x+3} < \frac{1}{x-2}$.

1. Rewrite the inequality in the form $f(x) < 0$, where $f(x)$ is a single rational expression.
2. Find the critical values for the inequality.
3. Hence, solve the inequality $\frac{2}{x+3} < \frac{1}{x-2}$.

Answer:

1. To rewrite the inequality, we move all terms to one side and combine them into a single fraction:

$$\begin{aligned}\frac{2}{x+3} - \frac{1}{x-2} &< 0\\ \frac{2(x-2) - 1(x+3)}{(x+3)(x-2)} &< 0\\ \frac{2x-4-x-3}{(x+3)(x-2)} &< 0\\ \frac{x-7}{(x+3)(x-2)} &< 0\end{aligned}$$

So, $f(x) = \frac{x-7}{(x+3)(x-2)}$.

2. The critical values are the values of x for which the numerator or the denominator is zero.
   - Numerator: $x - 7 = 0 \implies x = 7$.
   - Denominator: $(x+3)(x-2) = 0 \implies x = -3$ or $x = 2$.

   The critical values are $x = -3$, $x = 2$, and $x = 7$.

3. We use a sign table to determine the intervals where $f(x) < 0$.

x	$-\infty$		-3		2		7		$+\infty$
$x+3$		$-$	‖	$+$		$+$		$+$	
$x-2$		$-$		$-$	‖	$+$		$+$	
$x-7$		$-$		$-$		$-$	0	$+$	
$f(x)$		$-$	‖	$+$	‖	$-$	0	$+$	

We are looking for intervals where $f(x) < 0$. From the table, these are the intervals $(-\infty, -3)$ and $(2, 7)$.
The inequality is strict ($<$), so the critical values are not included in the solution.
Therefore, the solution is $(-\infty, -3) \cup (2, 7)$.

Ex 30: Let $P(x) = x^3 - 2x^2 - 5x + 6$.

1. Show that $(x - 1)$ is a factor of $P(x)$.
2. Hence, fully factorize $P(x)$.
3. Using the factors of $P(x)$, solve the inequality $x^3 - 2x^2 - 5x + 6 > 0$.

Answer:

1. To show that $(x - 1)$ is a factor, we can use the Factor Theorem and evaluate $P(1)$:

$$P(1) = (1)^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$$

   Since $P(1) = 0$, $(x - 1)$ is a factor of $P(x)$.

2. We can now use long division to find the other factors. Dividing $P(x)$ by $(x - 1)$:

$$\begin{array}{r|l} & x^2 \quad - x - 6 \\ \hline x-1 & x^3 - 2x^2 - 5x + 6 \\ & -x^3 + x^2 \\ \hline & \quad -x^2 - 5x \\ & \quad\ \ x^2 - x \\ \hline & \qquad -6x + 6 \\ & \qquad\ \ 6x - 6 \\ \hline & \qquad\qquad 0 \end{array}$$

   The quotient is $x^2 - x - 6$. We can factor this quadratic:

$$x^2 - x - 6 = (x - 3)(x + 2)$$

   Therefore, the full factorization of $P(x)$ is $(x-1)(x+2)(x-3)$.

3. The inequality $x^3 - 2x^2 - 5x + 6 > 0$ is equivalent to $(x+2)(x-1)(x-3) > 0$. The critical values are the roots of the polynomial: $x = -2$, $x = 1$, and $x = 3$. We use a sign table to solve the inequality.

x	$-\infty$		-2		1		3		$+\infty$
$x+2$		$-$	0	$+$		$+$		$+$	
$x-1$		$-$		$-$	0	$+$		$+$	
$x-3$		$-$		$-$		$-$	0	$+$	
$P(x)$		$-$	0	$+$	0	$-$	0	$+$	

We are looking for the intervals where $P(x) > 0$. From the table, these intervals are where the sign is positive. The inequality is strict, so the endpoints are not included. The solution is $(-2, 1) \cup (3, \infty)$.