GRAPH THEORY

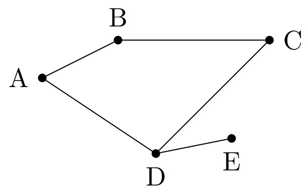
Graphs are used to show how things are connected. They can be used to help solve problems in train scheduling, traffic flow, bed usage in hospitals, and project management.
The theory of graphs was developed centuries ago, but the application of the theoretical ideas is relatively recent. Real progress was made during and after the Second World War as mathematicians, industrial technicians, and members of the armed services worked together to improve military operations.

A DEFINITIONS

Definition Graph

A **graph** (or network) is a structure which shows the physical connections or relationships between things of interest.
The things of interest are represented by **vertices** (singular: vertex), also called nodes or points.
The connections or relationships are represented by **edges**, also called lines or arcs.

Ex:

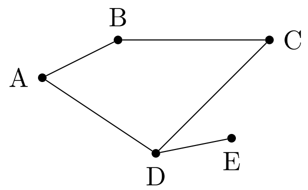


This graph shows road connections between several towns. We can see that towns A and B are directly connected, but towns A and C are not.

Definition Degree of a Vertex

The **degree** of a vertex, denoted $\deg(v)$, is the number of edges connected to it.
If a loop exists (an edge connecting a vertex to itself), it contributes 2 to the degree of that vertex.

Ex:



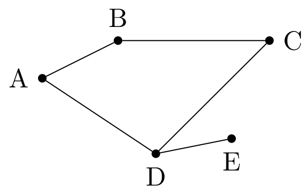
For vertex A: It has one connection to B and one connection to D. $\deg(A) = 2$.

Definition Undirected Graph

In an **undirected graph**, movement is allowed in either direction along the edges. The relationship is symmetric.

- **Adjacent vertices** are vertices connected by an edge.
- **Adjacent edges** are edges which share a common vertex.

Ex:



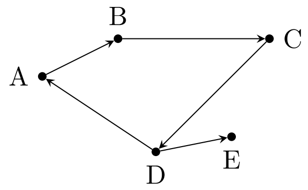
In this graph, A is connected to B and B is connected to A.

Definition Directed Graph

A **directed graph** (or digraph) contains arrows indicating the direction we can move along the edges.

- The **in-degree** is the number of edges coming into the vertex.
- The **out-degree** is the number of edges going out from the vertex.

Ex:

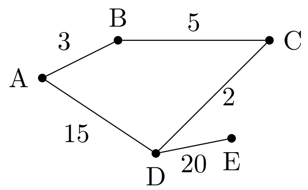


In this graph, A is connected to B and B is **not** connected to A.

Definition Weighted Graph

A **weighted graph** has numbers assigned to its edges. These numbers may represent the cost, time, or distance.

Ex:

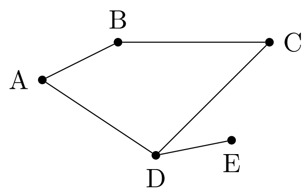


The weight of edge AD is 15.

Definition Walk, Trail, Path, circuit and Cycle

- A **walk** is a sequence of vertices such that each successive pair is adjacent.
- A **trail** is a walk in which no edge is repeated.
- A **path** is a walk in which no vertex is repeated.
- A **circuit** is a walk in which no edge is repeated that starts and ends at the same vertex.
- A **cycle** is a walk in which no vertex is repeated except the first/last vertex that starts and ends at the same vertex.

Ex:



In this graph:

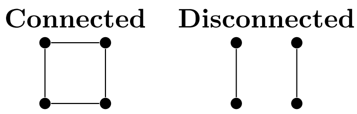
- $A \to B \to C$ is a **walk** because there is an edge between A and B and an edge between B and C.
- $A \to B \to A$ is not a **trail** because the edge between A and B is repeated.
- $A \to B \to C \to D \to A$ is not a **path** because the vertex A is repeated.
- $A \to B \to C \to D \to A$ is a **circuit** (and a cycle).

B PROPERTIES OF GRAPHS

Definition Connected Graph

A graph is called **connected** if it is possible to travel from any vertex to any other vertex via a sequence of edges.

Ex:

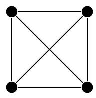


In the graph on the right, you cannot get from the left pair of vertices to the right pair. It has two disconnected components.

Definition Complete Graph

A **complete graph** is a simple undirected graph in which every vertex is connected by an edge to every other vertex. The notation K_n is used to describe the complete graph with n vertices.

Ex:



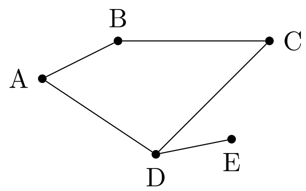
This is K_4. It has 4 vertices and $\frac{4(3)}{2} = 6$ edges.

C ADJACENCY MATRICES

Definition Adjacency Matrix

For a graph with n vertices, the **adjacency matrix** $\mathbf{M}$ is an $n \times n$ matrix where the element m_{ij} represents the number of direct edges from vertex i to vertex j.

Ex:



The table below shows the number of 1-step walks between the vertices:

		Finishing vertex				
		A	B	C	D	E
Starting vertex	A	0	1	0	1	0
	B	1	0	1	0	0
	C	0	1	0	1	0
	D	1	0	1	0	1
	E	0	0	0	1	0

The adjacency matrix for the graph is therefore:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since we do not usually write the labels for the vertices on the adjacency matrix, it is important to remember what the rows and columns refer to. To make this easier, we put them in alphabetical order.

Proposition Multi-step Walks

If $\mathbf{M}$ is the adjacency matrix, then the element (i, j) of $\mathbf{M}^k$ gives the number of walks of length k from vertex i to vertex j.

Ex: To understand *why* the powers of **M** give the multi-step adjacency matrices, consider the 2-step matrix:

$$\mathbf{M}^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The element in row 2 column 4 of $\mathbf{M}^2$ shows there are **two** 2-step walks from B to D.
This value comes from the multiplication **row 2** $\times$ **column 4** as follows:

- $1 \times 1 = 1$ walk from B to A $\times$ 1 walk from A to D
- $0 \times 0 = 0$ walks from B to B $\times$ 0 walks from B to D
- $1 \times 1 = 1$ walk from B to C $\times$ 1 walk from C to D
- $0 \times 0 = 0$ walks from B to D $\times$ 0 walks from D to D
- $0 \times 1 = 0$ walks from B to E $\times$ 1 walk from E to D

Adding these terms gives the two 2-step walks from B to D. The walks are B $\to$ A $\to$ D and B $\to$ C $\to$ D.