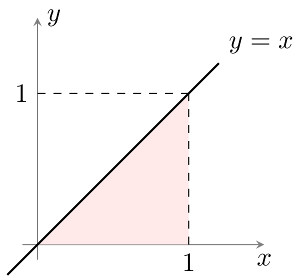
A APPROXIMATING AREA WITH RIEMANN SUMS

A.1 ESTIMATING AREA WITH LEFT AND RIGHT SUMS

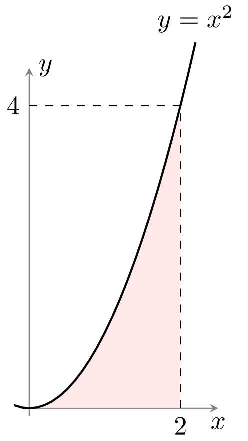
Ex 1: Consider the area between the curve of the function $f(x) = x$ and the x-axis from $x = 0$ to $x = 1$.

1. Divide the interval into 5 subintervals of equal width. Then, estimate the area by summing the areas of rectangles whose heights are determined by the function's value at:
   (a) the left-hand endpoint of each subinterval.
   (b) the right-hand endpoint of each subinterval.
2. Calculate the actual area and compare it with your estimations.



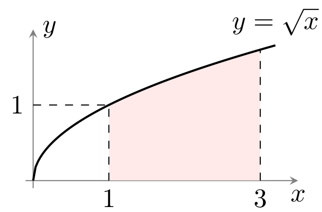
Ex 2: Consider the area, $\mathcal{A}$, under the curve of the function $f(x) = x^2$ from $x = 0$ to $x = 2$.

1. Divide the interval into 4 subintervals of equal width. Then, estimate the area by summing the areas of rectangles whose heights are determined by the function's value at:
   (a) the left-hand endpoint of each subinterval (L_4).
   (b) the right-hand endpoint of each subinterval (R_4).
2. By observing your diagrams, state whether your estimations are overestimates or underestimates, and write an inequality that relates L_4, R_4, and the true area $\mathcal{A}$.



Ex 3: Consider the area, $\mathcal{A}$, under the curve of the function $f(x) = \sqrt{x}$ from $x = 1$ to $x = 3$.

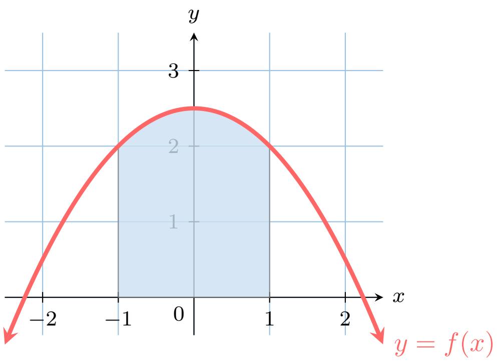
1. Divide the interval into 4 subintervals of equal width. Then, estimate the area by summing the areas of rectangles whose heights are determined by the function's value at:
   (a) the left-hand endpoint of each subinterval (L_4).
   (b) the right-hand endpoint of each subinterval (R_4).
2. By observing the function's behavior, state whether your estimations are overestimates or underestimates, and write an inequality that relates L_4, R_4, and the true area $\mathcal{A}$.



B DEFINITION OF THE DEFINITE INTEGRAL

B.1 IDENTIFYING THE DEFINITE INTEGRAL FOR A GIVEN AREA

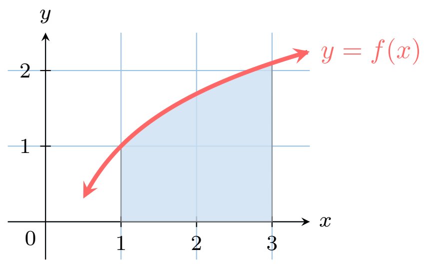
MCQ 4:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-1}^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-1}^1 f(x)\,\mathrm{d}x$

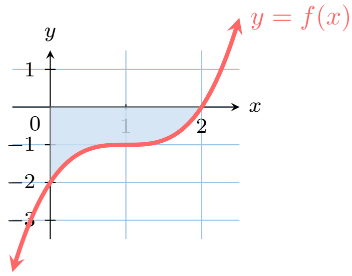
MCQ 5:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_1^3 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^3 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_1^2 f(x)\,\mathrm{d}x$

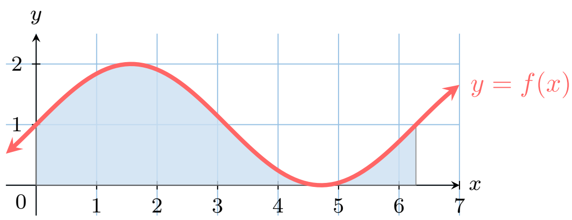
MCQ 6:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_1^2 f(x)\,\mathrm{d}x$

MCQ 7:

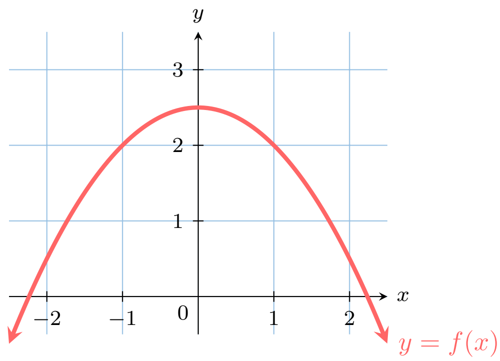


The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^{\pi} f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^{2\pi} f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$

B.2 INTERPRETING THE SIGN OF A DEFINITE INTEGRAL

MCQ 8:

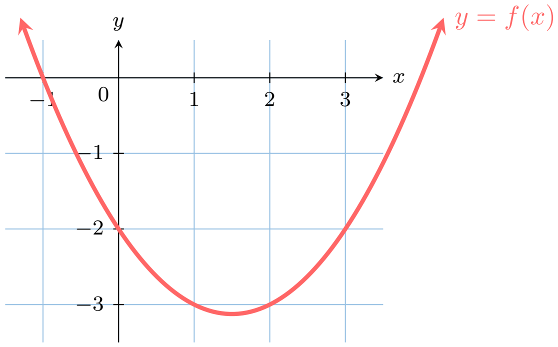


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\displaystyle\int_{-1}^1 f(x)\,\mathrm{d}x$.

☐ Positive

☐ Negative

MCQ 9:

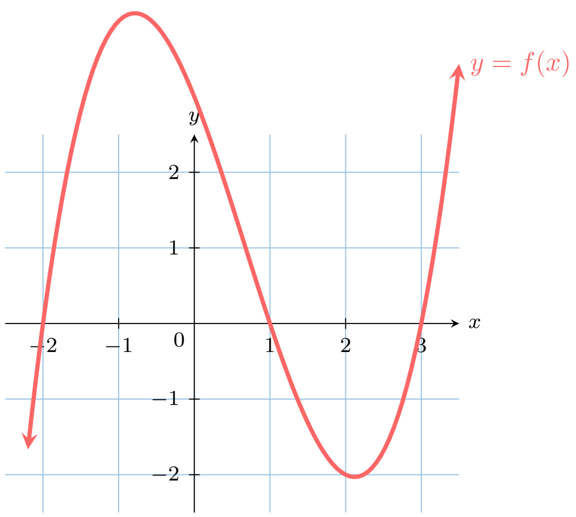


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_{0}^{3} f(x)\,\mathrm{d}x$.

☐ Positive

☐ Negative

MCQ 10:

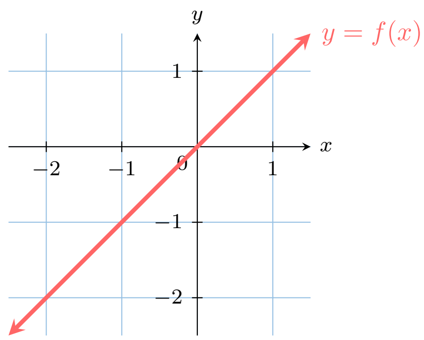


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_{-2}^{3} f(x)\,\mathrm{d}x$.

☐ Positive

☐ Negative

MCQ 11:



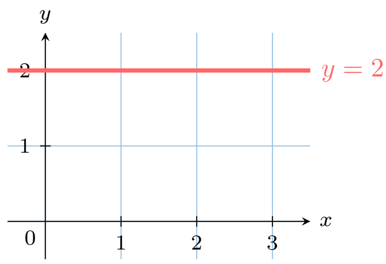
Considering the graph of the function $f(x) = x$ above, determine the sign of the definite integral $\int_{-2}^{1} f(x)\,\mathrm{d}x$.

☐ Positive

☐ Negative

B.3 EVALUATING INTEGRALS USING GEOMETRIC FORMULAS

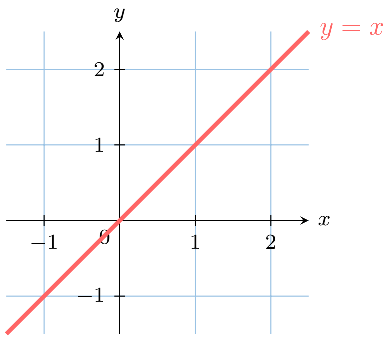
Ex 12:



Using the geometric interpretation of the integral as an area, find:

$$\int_{0}^{3} 2\,\mathrm{d}x = \square$$

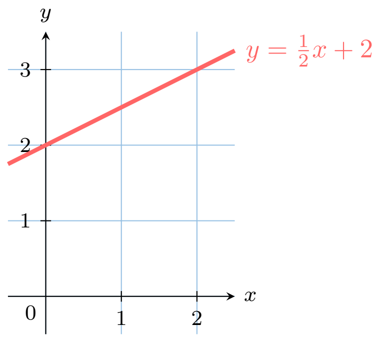
Ex 13:



Using the geometric interpretation of the integral as a signed area, find:

$$\int_{-1}^{2} x\,\mathrm{d}x = \square$$

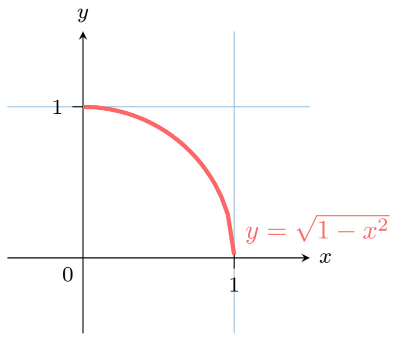
Ex 14:



Using the geometric interpretation of the integral as an area, find:

$$\int_{0}^{2} \left(\frac{1}{2}x + 2\right)\,\mathrm{d}x = \square$$

Ex 15:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \square$$

C PROPERTIES OF THE DEFINITE INTEGRAL

C.1 APPLYING THE PROPERTIES OF DEFINITE INTEGRALS

Ex 16: For a function f, $\int_0^1 f(x)\,\mathrm{d}x = 2$ and $\int_1^2 f(x)\,\mathrm{d}x = 1$, find:

$$\int_0^2 f(x)\,\mathrm{d}x = \square$$
$$\int_0^0 f(x)\,\mathrm{d}x = \square$$
$$\int_0^2 4f(x)\,\mathrm{d}x = \square$$

Ex 17: Given that $\int_1^3 f(x)\,\mathrm{d}x = 4$ and $\int_1^3 g(x)\,\mathrm{d}x = -2$, find:

$$\int_1^3 (f(x)+g(x))\,\mathrm{d}x = \square$$
$$\int_1^3 (2f(x)-3g(x))\,\mathrm{d}x = \square$$

Ex 18: Given that $\int_0^3 f(x)\,\mathrm{d}x = -5$ and $\int_0^1 f(x)\,\mathrm{d}x = 2$, find the value of $\int_1^3 f(x)\,\mathrm{d}x$.

$$\int_1^3 f(x)\,\mathrm{d}x = \square$$

Ex 19: Given that $\int_2^5 f(x)\,\mathrm{d}x = 10$ and $\int_2^5 g(x)\,\mathrm{d}x = 3$, find:

$$\int_2^5 (f(x)-g(x))\,\mathrm{d}x = \square$$
$$\int_2^5 5g(x)\,\mathrm{d}x = \square$$

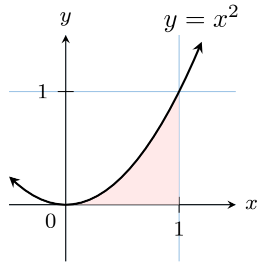
Ex 20: Given that $\int_{-1}^4 h(x)\,\mathrm{d}x = 6$ and $\int_2^4 h(x)\,\mathrm{d}x = 5$, find the value of $\int_{-1}^2 h(x)\,\mathrm{d}x$.

$$\int_{-1}^2 h(x)\,\mathrm{d}x = \square$$

D FUNDAMENTAL THEOREM OF CALCULUS

D.1 CALCULATING AREA USING THE FUNDAMENTAL THEOREM

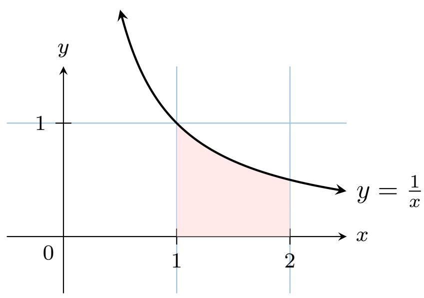
Ex 21:



Find the area of the region enclosed by the x-axis, the curve $y = x^2$, and the lines $x = 0$ and $x = 1$.

Area = $\square$ units^2

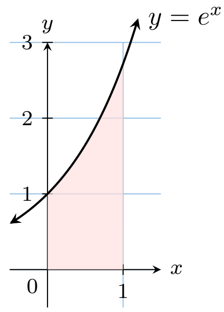
Ex 22:



Find the area of the region enclosed by the x-axis, the curve $y = \frac{1}{x}$, and the lines $x = 1$ and $x = 2$.

Area = $\square$ units^2

Ex 23:



Find the area of the region enclosed by the x-axis, the curve $y = e^x$, and the lines $x = 0$ and $x = 1$.

Area = $\square$ units^2

D.2 EVALUATING DEFINITE INTEGRALS: LEVEL 1

Ex 24: Find the value of the definite integral:

$$\int_0^3 x\,dx = \square$$

Ex 25: Find the value of the definite integral:

$$\int_0^\pi \sin(x)\,dx = \square$$

Ex 26: Find the value of the definite integral:

$$\int_0^2 e^x\,dx = \square$$

Ex 27: Find the value of the definite integral:

$$\int_1^e \frac{1}{x}\,dx = \square$$

D.3 EVALUATING DEFINITE INTEGRALS: LEVEL 2

Ex 28: Find the value of the definite integral:

$$\int_1^2 (3x^2 + 2x - 1)\,dx = \square$$

Ex 29: Find the value of the definite integral:

$$\int_{\pi/2}^{\pi} (2\sin(x) + \cos(x))\,dx = \square$$

Ex 30: Find the value of the definite integral:

$$\int_1^3 \frac{6}{x^3}\,dx = \square$$

Ex 31: Find the value of the definite integral:

$$\int_0^1 2xe^{x^2}\,dx = \square$$

D.4 DEFINING FUNCTIONS USING DEFINITE INTEGRALS

Ex 32: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_{\pi/2}^{x} \cos(t)\,dt$$

$$F(x) = \square$$

Ex 33: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0$$

$$F(x) = \square$$

Ex 34: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_0^x (u^2 + 1)\,du$$

$$F(x) = \square$$

D.5 PROVING THE PROPERTIES OF DEFINITE INTEGRALS

Ex 35: Using the fundamental theorem of calculus, prove that

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

Ex 36: Using the fundamental theorem of calculus, prove Chasles's Relation:

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Ex 37: Using the fundamental theorem of calculus, prove that

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Ex 38: Using the fundamental theorem of calculus, prove that

$$\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Ex 39: Using the fundamental theorem of calculus, prove that for any constant k:

$$\int_a^b k f(x)\,\mathrm{d}x = k \int_a^b f(x)\,\mathrm{d}x$$

D.6 STUDYING SEQUENCES DEFINED BY INTEGRALS

Ex 40: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 x^n\,dx$$

1. Calculate the first three terms of the sequence: u_0, u_1, and u_2.
2. Find a general formula for u_n.

- $u_0 =$
- $u_1 =$
- $u_2 =$
- $u_n =$

Ex 41: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 \frac{x^n}{1+x}\,dx$$

1. Calculate u_0.
2. Prove that for any integer $n \geq 0$, the recurrence relation $u_{n+1} + u_n = \frac{1}{n+1}$ holds.
3. Hence, deduce the value of u_1.

Ex 42: A sequence (u_n) is defined for any integer $n > 0$ by the integral:

$$u_n = \int_0^1 \frac{e^{nx}}{1+e^x}\,dx$$

1. Calculate u_1.
2. Prove that for any integer $n > 0$, the following recurrence relation holds:
$$u_{n+1} + u_n = \frac{e^n - 1}{n}$$
3. Hence, deduce the value of u_2.

E INTEGRATION BY PARTS

E.1 EVALUATING DEFINITE INTEGRALS BY PARTS

Ex 43: Find the value of the definite integral $\displaystyle\int_0^1 xe^x\,dx$.

$$\int_0^1 xe^x\,dx =$$

Ex 44: Find the value of the definite integral $\displaystyle\int_0^{\pi/2} x\sin(x)\,dx$.

$$\int_0^{\pi/2} x\sin(x)\,dx = \square$$

Ex 45: Find the value of the definite integral $\int_1^e x\ln(x)\,dx$.

$$\int_1^e x\ln(x)\,dx = \square$$

Ex 46: Find the value of the definite integral $\int_0^{\pi} x\cos(x)\,dx$.

$$\int_0^{\pi} x\cos(x)\,dx = \square$$

E.2 APPLYING ADVANCED INTEGRATION TECHNIQUES

Ex 47: Find the value of the definite integral $\int_0^1 x^2e^x\,dx$.

$$\int_0^1 x^2e^x\,dx = \square$$

F INTEGRALS AND INEQUALITIES

F.1 DETERMINING THE SIGN OF AN INTEGRAL

Ex 48: Determine the sign of the integral $\int_{-2}^{-1}\frac{1}{x}\,dx$ without calculating it.

Ex 49: Determine the sign of the integral $\int_{-3}^{-1}(2x^2+1)\,dx$ without calculating it.

Ex 50: Determine the sign of the integral $\int_0^1 2xe^x\,dx$ without calculating it.

Ex 51: Determine the sign of the integral $\int_{1/2}^1 \ln(x)\,dx$ without calculating it.

F.2 INTEGRATING INEQUALITIES

Ex 52: Let f be the function defined on $\mathbb{R}$ by $f(x)=e^{-x^2}$.

1. Show that for all real numbers $x\geq 1$, $0\leq f(x)\leq e^{-x}$.
2. Deduce a bound for the integral $\int_1^2 f(x)\,dx$.

Ex 53: Let f be the function defined on $\mathbb{R}$ by $f(x)=\frac{1}{x^2+1}$.

1. Show that for all real numbers $x\in[0,2]$, $\frac{1}{5}\leq f(x)\leq 1$.
2. Deduce a bound for the integral $\int_0^2 f(x)\,dx$.

F.3 CALCULATING AND INTERPRETING MEAN VALUES

Ex 54: Consider the function $f : x \mapsto \dfrac{1}{\sqrt{x}}$.
Determine the mean value μ of f on the interval $[1, 4]$ and provide a geometric interpretation of the result.

Ex 55: Consider the function $f : x \mapsto x^2 + 3$.
Determine the mean value μ of f on the interval $[-2, 2]$ and provide a geometric interpretation of the result.

Ex 56: Consider the function $f : x \mapsto \dfrac{2x}{x^2+1}$.
Determine the mean value μ of f on the interval $[0, 1]$ and provide a geometric interpretation of the result.

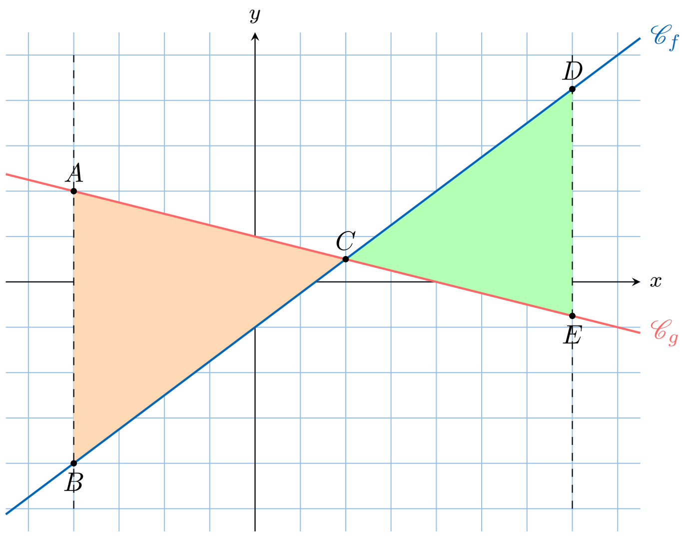
G AREA BETWEEN TWO CURVES

G.1 CALCULATING THE AREA BETWEEN TWO CURVES

Ex 57: Consider two functions f and g defined on $\mathbb{R}$ by:

$$f(x) = \frac{3}{4}x - 1 \quad \text{and} \quad g(x) = -\frac{1}{4}x + 1$$

The curves $\mathscr{C}_f$ and $\mathscr{C}_g$ are shown below, intersecting at point $C(2, 0.5)$. Two regions are shaded: triangle ABC (orange) and triangle CDE (green).



1. Solve $f(x) \geq g(x)$ by determining the interval of x for which the inequality is true.
2. Express the area of triangle ABC using an integral and calculate its value.
3. Express the area of triangle CDE using an integral and calculate its value.

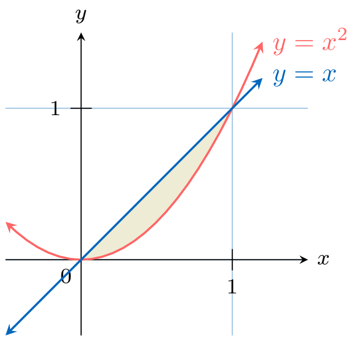
Ex 58: Consider the functions f and g defined on $\mathbb{R}$ by:

$$f(x) = 1 - x^2 \quad \text{and} \quad g(x) = x^2 - 1$$

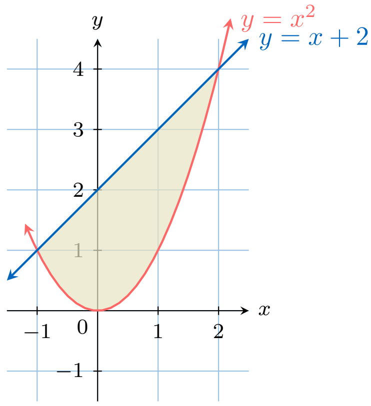
Calculate the area of the region bounded by these two curves between $x = -1$ and $x = 1$.

G.2 CALCULATING THE AREA ENCLOSED BETWEEN TWO CURVES

Ex 59: Find the area of the finite region enclosed between the curves $y = x$ and $y = x^2$.



Ex 60: Find the area of the finite region enclosed between the curves $y = x + 2$ and $y = x^2$.



Ex 61: Find the area of the finite region enclosed between the curves $y = 2x - x^2$ and $y = x^2 - 4x$.