INTERESTS

A DEFINITIONS

Discover: Understanding Interest: Simple vs. Compound Interest We've all heard of interest rates—whether on a mortgage, a credit card, or a loan. But what does it really mean?
Interest is essentially the "rent" you pay for borrowing money. It's the additional amount you pay to use someone else's money for a certain period of time.

- **Example of interest:** Imagine you borrow \$100 from someone today and promise to pay it back in one year. If you return exactly \$100 after one year, there's no interest involved. However, the lender might ask for more in return—they might want to be compensated for letting you use their money.
  They may request a percentage of the amount. For example, at a 10% interest rate per year, the interest you would pay is:
  $$\begin{aligned}\text{Interest Paid} &= \text{Percentage of Original Amount}\\ &= \text{Interest Rate} \times \text{Original Amount}\\ &= 10\% \times 100\\ &= \frac{10}{100} \times 100\\ &= 10 \text{ dollars}\end{aligned}$$
  Therefore, after one year, you would owe:
  $$\begin{aligned}\text{Amount at Year 1} &= \text{Original Amount} + \text{Interest Paid}\\ &= 100 + 10\\ &= 110 \text{ dollars}\end{aligned}$$
  In this case, you would pay back \$110 instead of \$100. The extra \$10 is the interest, which represents the cost of borrowing the money for a year.
- **Simple interest:** Suppose you borrow \$100 at a 10% interest rate per year. With simple interest, the interest is calculated only on the percentage of the original amount every year.
  - $\begin{aligned}\text{Amount at Year 1} &= \text{Original Amount} + \text{Percentage of Original Amount}\\ &= 100 + \frac{10}{100} \times 100\\ &= 100 + 10\\ &= 110 \text{ dollars}\end{aligned}$
  - $\begin{aligned}\text{Amount at Year 2} &= \text{Amount at Year 1} + \text{Percentage of Original Amount}\\ &= 110 + \frac{10}{100} \times 100\\ &= 110 + 10\\ &= 120 \text{ dollars}\end{aligned}$
  - $\begin{aligned}\text{Amount at Year 3} &= \text{Amount at Year 2} + \text{Percentage of Original Amount}\\ &= 120 + \frac{10}{100} \times 100\\ &= 120 + 10\\ &= 130 \text{ dollars}\end{aligned}$
  - $\vdots$
  - $\begin{aligned}\text{Amount at Year 10} &= \text{Amount at Year 9} + \text{Percentage of Original Amount}\\ &= 190 + \frac{10}{100} \times 100\\ &= 190 + 10\\ &= 200 \text{ dollars}\end{aligned}$

  After 10 years, you would owe \$200, which includes \$100 of the original amount and \$100 in interest (\$10 per year for 10 years).
- **Compound interest:** Unlike simple interest, compound interest is calculated on the original amount plus any accumulated interest. This means you pay interest on both the initial amount and the interest from previous periods.

– Amount after Year 1 = Original Amount + Percentage of Original Amount

$$= 100 + \frac{10}{100} \times 100$$
$$= 100 + 10$$
$$= 110 \text{ dollars}$$

– Amount after Year 2 = Amount after Year 1 + Percentage of Amount after Year 1

$$= 110 + \frac{10}{100} \times 110$$
$$= 110 + 11$$
$$= 121 \text{ dollars}$$

– Amount after Year 3 = Amount after Year 2 + Percentage of Amount after Year 2

$$= 121 + \frac{10}{100} \times 121$$
$$= 121 + 12.10$$
$$= 133.10 \text{ dollars}$$

– ⋮

– Amount after Year 10 = Amount after Year 9 + Percentage of Amount after Year 9

$$= 235.79 + \frac{10}{100} \times 235.79$$
$$= 235.79 + 23.58$$
$$= 259.37 \text{ dollars}$$

After 10 years, you would owe \$259.37, which includes \$100 of the original amount and \$159.37 in compounded interest.

- **Conclusion:** As you can see, with compound interest, the amount you owe grows faster than with simple interest because you are paying interest on the interest. The key takeaway is that **simple interest** is calculated on the original amount every year, while **compound interest** is calculated on the original amount plus any accumulated interest.

Definition Principal

The **principal** is the original amount of money that is either invested or loaned.

Definition Interest

Interest is the cost paid for borrowing money or the amount earned from lending or investing money.

B SIMPLE INTEREST

Discover: Suppose you borrow \$100 with an interest rate of 10% per year. With simple interest, the interest is calculated only on the initial amount each year.

- Total interest paid after 1 year = Percentage of Original Amount

$$= \text{Interest Rate} \times \text{Original Amount}$$
$$= 10\% \times 100$$
$$= \frac{10}{100} \times 100$$
$$= 10 \text{ dollars}$$

- Total interest paid after 2 years = 2 × Percentage of Original Amount

$$= 2 \times \text{Interest Rate} \times \text{Original Amount}$$
$$= 2 \times 10\% \times 100$$
$$= 2 \times \frac{10}{100} \times 100$$
$$= 20 \text{ dollars}$$

- Total interest paid after 3 years = 3 × Percentage of Original Amount

$$= 3 \times \text{Interest Rate} \times \text{Original Amount}$$
$$= 3 \times 10\% \times 100$$
$$= 3 \times \frac{10}{100} \times 100$$
$$= 30 \text{ dollars}$$

These observations lead to the simple interest formula:

$$\text{Simple Interest} = \text{Number of years} \times \text{Interest rate} \times \text{Principal (initial amount)}$$

Definition Simple Interest

The **simple interest** is calculated each year as a fixed percentage on the principal (original amount) of money borrowed or invested.

Proposition Simple Interest Formula

The simple interest, denoted by I, is calculated as:

$$I = t \times r \times P$$

where:

- P is the principal (original amount)
- r is the interest rate per year
- t is the time (in years)

The final amount, denoted by A, is:

$$\begin{aligned} A &= P + I \\ &= P + t \times r \times P \\ &= (1 + t \times r) \times P \end{aligned}$$

Ex: Find the simple interest on a principal of \$500 at a rate of 3% per year over 5 years.

Answer:

$$\begin{aligned} \text{Interest} &= 5 \times 3\% \text{ of } 500 \\ &= 5 \times \frac{3}{100} \times 500 \\ &= 75 \text{ dollars} \end{aligned}$$

Method Finding the Principal

To find the principal P, use the formula:

$$P = \frac{I}{t \times r}$$

Proof

We can use the simple interest formula to find the original amount (principal):

$$\text{Interest} = \text{Time} \times \text{Rate} \times \text{Principal}$$

Rearranging to solve for the principal:

$$\text{Principal} = \frac{\text{Interest}}{\text{Time} \times \text{Rate}}$$

Ex: An investment earns \$100 over 5 years at a rate of 5% per year. Find the principal.

Answer:

$$\begin{aligned} P &= \frac{I}{t \times r} \\ &= \frac{100}{5 \times \frac{5}{100}} \\ &= 400 \text{ dollars} \end{aligned}$$

The principal is \$400.

Method Finding the Rate

To find the rate r, use the formula:

$$r = \frac{I}{t \times P}$$

Proof

We can use the simple interest formula to find the interest rate:

$$\text{Interest} = \text{Time} \times \text{Rate} \times \text{Principal}$$

Rearranging to solve for the rate:

$$\text{Rate} = \frac{\text{Interest}}{\text{Time} \times \text{Principal}}$$

Ex: A principal of \$500 earns \$60 over 5 years. Find the interest rate per year.

Answer:

$$\begin{aligned} r &= \frac{I}{t \times P} \\ &= \frac{60}{5 \times 500} \\ &= 0.024 \\ &= \frac{2.4}{100} \\ &= 2.4\% \end{aligned}$$

The interest rate per year is 2.4%.

Method Finding the Number of Years

To find the number of years t, use the formula:

$$t = \frac{I}{r \times P}$$

Proof

We can use the simple interest formula to find the time:

$$\text{Interest} = \text{Time} \times \text{Rate} \times \text{Principal}$$

Rearranging to solve for the time:

$$\text{Time} = \frac{\text{Interest}}{\text{Rate} \times \text{Principal}}$$

Ex: A principal of \$1 000 earns \$300 at an interest rate of 3% per year. Find the number of years.

Answer:

$$\begin{aligned} t &= \frac{I}{r \times P} \\ &= \frac{300}{\frac{3}{100} \times 1\,000} \\ &= 10 \text{ years} \end{aligned}$$

The number of years is 10.

C COMPOUND INTEREST

Discover: If you leave money in the bank for a period of time, the interest earned is automatically added to your account. After the interest is added, it also begins to earn interest in the next time period. This process is called compound interest.
Example of compound interest: \$1 000 is placed in an account that earns 10% interest per annum (p.a.), and the interest is allowed to compound over three years. This means the account is earning 10% p.a. in compound interest.
We can illustrate this in a table:

Year	Amount	Interest Earned
0	\$1 000	10% of \$1 000 = \$100
1	\$1 000 + \$100 = \$1 100	10% of \$1 100 = \$110
2	\$1 100 + \$110 = \$1 210	10% of \$1 210 = \$121
3	\$1 210 + \$121 = \$1 331	—

After 3 years, there will be a total of \$1 331 in the account, meaning we have earned \$331 in compound interest.
We can calculate the final amount using another method as well:

- $\text{Amount after 1 year} = \text{Initial amount} + \text{Interest on the initial amount}$
$$\begin{aligned} &= 1\,000 + 0.1 \times 1\,000 \\ &= (1 + 0.1) \times 1\,000 \quad \text{(factoring out } 1\,000\text{)} \\ &= 1.1 \times 1\,000 \end{aligned}$$
- $\text{Amount after 2 years} = \text{Amount after 1 year} + \text{Interest on the amount after 1 year}$
$$\begin{aligned} &= 1.1 \times 1\,000 + 0.1 \times 1.1 \times 1\,000 \\ &= (1 + 0.1) \times 1.1 \times 1\,000 \quad \text{(factoring out } 1.1 \times 1\,000\text{)} \\ &= 1.1^2 \times 1\,000 \end{aligned}$$
- $\text{Amount after 3 years} = \text{Amount after 2 years} + \text{Interest on the amount after 2 years}$
$$\begin{aligned} &= 1.1^2 \times 1\,000 + 0.1 \times 1.1^2 \times 1\,000 \\ &= (1 + 0.1) \times 1.1^2 \times 1\,000 \quad \text{(factoring out } 1.1^2 \times 1\,000\text{)} \\ &= 1.1^3 \times 1\,000 \end{aligned}$$

These observations lead to the compound interest formula:

$$\text{Final amount} = (1 + \text{Interest rate})^{\text{Number of years}} \times \text{Initial amount}$$

Definition Compound Interest

Compound interest is interest that accumulates on both the principal sum and the previously accumulated interest.

Proposition Annual Compound Interest Formula

The final amount of an investment with interest compounded annually is:

$$A = P(1 + r)^t$$

where:

- P is the principal,
- r is the annual interest rate,
- t is the time (in years).

Ex: Find the final amount for compound interest on a principal of \$500 at a rate of 3% per year over 5 years.

Answer:

$$\begin{aligned} A &= P(1 + r)^t \\ &= 500 \times (1 + 0.03)^5 \\ &\approx \$580.81 \end{aligned}$$

The final amount is approximately \$580.81.

D COMPOUND INTEREST BY PERIOD

Definition Compounding by Period

Interest can be compounded in various periods, including:

- monthly (12 times per year)
- semi-annually (2 times per year)
- weekly (52 times per year)

Proposition Periodic Compound Interest Formula

The final amount of an investment with interest compounded periodically is:

$$A = P\left(1 + \frac{r}{c_y}\right)^{c_y t}$$

where:

- P is the principal,

- r is the annual interest rate,
- t is the time (in years),
- c_y is the number of times the compound interest is applied per year.

Ex: Calculate the final amount of a \$5 000 principal invested at an annual interest rate of 2%, compounded monthly, over a period of 10 years.

Answer:

$$\begin{aligned} A &= P\left(1+\frac{r}{c_y}\right)^{c_y t} \\ &= 5\,000\left(1+\frac{0.02}{12}\right)^{12\times 10} \\ &\approx \$6\,106.71 \end{aligned}$$

The final amount is approximately \$6 106.71.

Method **Finding Any Value Using a Graphing Calculator**

The TVM Solver (Time Value of Money) can be used to find any variable if all the other variables are given.

- PV is the principal, considered as an outgoing, and is entered as a negative value ($PV = -P$).
- FV represents the final amount ($FV = A$).
- C/Y is the number of compounding periods per year ($C/Y = c_y$).
- n represents the number of compounding periods, not the number of years ($n = c_y \times t$).
- PMT and P/Y are not used in this case. $P/Y = C/Y$ and $PMT = 0$.

Ex: Find the final amount on a principal of \$23 000 at a rate of 3.45% over 6 years compounded quarterly.

Answer: Using the TVM Solver with:

$$n = 6\times 4 = 24,\quad I\% = 3.45,\quad PV = -23\,000,\quad C/Y = 4,\quad (PMT = 0,\quad P/Y = 4)$$

we find the final amount:

$$FV \approx \$28\,264.50$$

E VARIABLE RATE INVESTMENTS

Discover: For many investments, the interest rate can increase or decrease over time. These investments are known as variable rate investments.

Method **Calculation for Variable Rate Investments**

If the interest rate varies over the term of the investment, separate calculations must be performed for each interest rate period.

Ex: Louis invested \$200 in a variable rate investment account for 8 years. The interest rates were:

- for the first six years: 3% compounded yearly.
- for the last two years: 2% compounded yearly.

Find the final amount after 8 years.

Answer: We can solve this problem using two methods: the compound interest formula and the TVM Solver.

- **Method 1: Using the Compound Interest Formula**
  - First, calculate the amount after the first 6 years at 3% interest compounded yearly:

$$\begin{aligned} A_1 &= P\times(1+r)^{t_1} \\ &= 200\times(1+0.03)^6 \\ &= 200\times(1.03)^6 \\ &\approx 200\times 1.194052 \\ &\approx \$238.81 \end{aligned}$$

- Next, calculate the amount after the last 2 years at 2% interest compounded yearly, using A_1 as the principal:

$$\begin{aligned} A &= A_1 \times (1+r)^{t_2} \\ &= 238.81 \times (1+0.02)^2 \\ &= 238.81 \times (1.02)^2 \\ &= 238.81 \times 1.0404 \\ &\approx \$248.46 \end{aligned}$$

- Therefore, the final amount after 8 years is approximately \$248.46.

- **Method 2: Using the TVM Solver**
  - First period (first 6 years at 3% interest compounded yearly):
    * $PV = -200$ (initial investment)
    * $I\% = 3$
    * $n = 6$
    * $PMT = 0$
    * $FV = ?$
    * $P/Y = 1$
    * $C/Y = 1$

    Calculating FV, we get $FV \approx \$238.81$.
  - Second period (next 2 years at 2% interest compounded yearly):
    * $PV = -238.81$
    * $I\% = 2$
    * $n = 2$
    * $PMT = 0$
    * $FV = ?$
    * $P/Y = 1$
    * $C/Y = 1$

    Calculating FV, we get $FV \approx \$248.46$.
  - Thus, the final amount after 8 years is approximately \$248.46.

F PERIODIC PAYMENT

Discover: Calculating Periodic Payments When taking out a loan or mortgage, the borrower often agrees to repay the loan in regular intervals, known as **periodic payments**. These payments typically include both the principal (the original loan amount) and interest (the cost of borrowing). The amount of each periodic payment depends on factors such as the loan amount, the interest rate, the length of the loan, and the compounding frequency.
To calculate the periodic payment (PMT) for an amortized loan, we use the following formula:

$$PMT = \frac{P \times \frac{r}{c_y}}{1 - \left(1 + \frac{r}{c_y}\right)^{-c_y t}}$$

where:

- P is the principal (the loan amount).
- r is the annual interest rate, expressed as a decimal.
- c_y is the number of compounding periods per year.
- t is the total number of years for the loan term.
- PMT is the periodic payment (e.g., the monthly payment).

While financial professionals typically use specialized software to compute the periodic payment, in this lesson, we will use the TVM (Time Value of Money) solver on your calculator. This tool makes it easier to calculate loan payments by entering the appropriate loan variables.

Method Using a Graphing Calculator to Find Any Value

The TVM Solver can be used to find any unknown variable if the other variables are known. This tool is helpful when calculating the periodic payment, principal, or interest rate in financial problems.The TVM Solver works as follows:

- PV (Present Value) is the principal loan amount. It is considered an outgoing and is entered as a negative value ($PV = -P$).
- FV (Future Value) is the final amount, typically zero for loans ($FV = 0$).
- $I\%$ is the annual interest rate.
- n is the total number of payments ($n = c_y \times t$).
- PMT represents the periodic payment (what we aim to find in many cases).
- P/Y represents the number of payment periods per year. For monthly payments, $P/Y = 12$.
- C/Y represents the number of compounding periods per year ($C/Y = c_y$).

By inputting these values, you can use the TVM Solver to compute any unknown variable.

Ex: Maria takes out a loan of \$15 000 to buy a car. The loan has an interest rate of 5.5% per annum, compounded monthly, and she agrees to repay the loan over 5 years.
Calculate the monthly payment Maria needs to make using the TVM Solver on your calculator.

Answer: We will use the TVM Solver to find the monthly payment.

- $PV = -15\,000$ (the loan amount is an outgoing, so it's negative).
- $FV = 0$ (since there is no remaining balance at the end of the loan).
- $I\% = 5.5$ (the annual interest rate).
- $n = 12 \times 5 = 60$ (monthly payments over 5 years).
- $PMT =?$ (the value we are solving for).
- $P/Y = 12$ (monthly payments).
- $C/Y = 12$ (compounded monthly).

Entering these values into your calculator's TVM Solver, we calculate:

$$PMT \approx \$286.52$$

Therefore, Maria will need to make monthly payments of approximately \$286.52 to repay her car loan over 5 years.