LOGARITHMS

A DEFINITION

A.1 EVALUATING LOGARITHMS

Ex 1: Evaluate $\log 100 = \boxed{2}$.

Answer: $\log(100) = \log\left(10^2\right)$
$= 2$

Ex 2: Evaluate $\log 0.1 = \boxed{-1}$.

Answer: $\log(0.1) = \log\left(10^{-1}\right)$
$= -1$

Ex 3: Evaluate $\log\left(\frac{1}{100}\right) = \boxed{-2}$.

Answer:
$$\begin{aligned}\log\left(\frac{1}{100}\right) &= \log\left(\frac{1}{10^2}\right)\\ &= \log\left(10^{-2}\right)\\ &= -2\end{aligned}$$

Ex 4: Evaluate $\log\sqrt{10} = \boxed{0.5}$.

Answer: $\log(\sqrt{10}) = \log(10^{0.5})$
$= 0.5$

Ex 5: Evaluate $\log 1 = \boxed{0}$.

Answer: $\log(1) = \log\left(10^0\right)$
$= 0$

A.2 EVALUATING USING A CALCULATOR

Ex 6: Evaluate (round to 2 decimal places).

$$\log(2) \approx \boxed{0.30}$$

Answer: By entering $\log(2)$ and pressing the equal button, the calculator displays: 0.30103.So, $\log(2) \approx 0.30$ (rounded to two decimal places).

Ex 7: Evaluate (round to 2 decimal places).

$$\log(0.2) \approx \boxed{-0.70}$$

Answer: By entering $\log(0.2)$ and pressing the equal button, the calculator displays: -0.69897.
So, $\log(0.2) \approx -0.70$ (rounded to two decimal places).

Ex 8: Evaluate (round to 2 decimal places).

$$\log(2 \times 10^9) \approx \boxed{9.30}$$

Answer: By entering $\log(2 \times 10^9)$ and pressing the equal button, the calculator displays: 9.30103.
So, $\log(2 \times 10^9) \approx 9.30$ (rounded to two decimal places).

A.3 SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

Ex 9: Find x such that $8 = 10^x$.

$x \approx \boxed{0.903}$ (rounded to 3 decimal places)

Answer: To solve $8 = 10^x$, take the logarithm (base 10) of both sides:
$$\begin{aligned}8 &= 10^x\\ \log(8) &= \log(10^x)\\ \log(8) &= x\\ x &\approx 0.903\end{aligned}$$
So, $x \approx 0.903$ (rounded to 3 decimal places).

Ex 10: Find x such that $0.4 = 10^x$.

$x \approx \boxed{-0.398}$ (rounded to 3 decimal places)

Answer: To solve $0.4 = 10^x$, take the logarithm (base 10) of both sides:
$$\begin{aligned}0.4 &= 10^x\\ \log(0.4) &= \log(10^x)\\ \log(0.4) &= x\\ x &\approx -0.398\end{aligned}$$
So, $x \approx -0.398$ (rounded to 3 decimal places).

Ex 11: Find x such that $250 = 10^x$.

$x \approx \boxed{2.398}$ (rounded to 3 decimal places)

Answer: To solve $250 = 10^x$, take the logarithm (base 10) of both sides:
$$\begin{aligned}250 &= 10^x\\ \log(250) &= \log(10^x)\\ \log(250) &= x\\ x &\approx 2.398\end{aligned}$$
So, $x \approx 2.398$ (rounded to 3 decimal places).

A.4 SOLVING FOR x WHEN $\log(x)$ IS GIVEN

Ex 12: Find x such that $\log(x) = 3$.

$$x = \boxed{1000}$$

Answer: Take $10^{\cdot}$ on both sides:
$$\begin{aligned}\log(x) &= 3\\ 10^{\log(x)} &= 10^3\\ x &= 10^3\\ x &= 1000\end{aligned}$$

Ex 13: Find x such that $\log(x) = -1$.

$$x = \boxed{0.1}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned}\log(x) &= -1\\ 10^{\log(x)} &= 10^{-1}\\ x &= 10^{-1}\\ x &= 0.1\end{aligned}$$

Ex 14: Find x such that $\log(x) = 0$.

$$x = \boxed{1}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned}\log(x) &= 0\\ 10^{\log(x)} &= 10^{0}\\ x &= 10^{0}\\ x &= 1\end{aligned}$$

Ex 15: Find x such that $\log(x) = 7$.

$$x = \boxed{10000000}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned}\log(x) &= 7\\ 10^{\log(x)} &= 10^{7}\\ x &= 10^{7}\\ x &= 10\,000\,000\end{aligned}$$

B LAWS OF LOGARITHMS

B.1 WRITING AS A SINGLE LOGARITHM: LEVEL 1

Ex 16: Write as a single logarithm

$$\log(5) + \log(3) = \boxed{\log(15)}$$

Answer:
$$\begin{aligned}\log(5) + \log(3) &= \log(5 \times 3)\\ &= \log 15\end{aligned}$$

Ex 17: Write as a single logarithm in the form $\log k$:

$$\log(15) - \log(5) = \boxed{\log(3)}$$

Answer:
$$\begin{aligned}\log(15) - \log(5) &= \log\left(\frac{15}{5}\right)\\ &= \log(3)\end{aligned}$$

Ex 18: Write as a single logarithm in the form $\log k$:

$$\log(4) + \log\left(\frac{1}{2}\right) = \boxed{\log(2)}$$

Answer:
$$\begin{aligned}\log(4) + \log\left(\frac{1}{2}\right) &= \log\left(4 \times \frac{1}{2}\right)\\ &= \log(2)\end{aligned}$$

Ex 19: Write as a single logarithm in the form $\log k$:

$$\log(18) - \log(3) = \boxed{\log(6)}$$

Answer:
$$\begin{aligned}\log(18) - \log(3) &= \log\left(\frac{18}{3}\right)\\ &= \log(6)\end{aligned}$$

B.2 WRITING AS A SINGLE LOGARITHM: LEVEL 2

Ex 20: Write as a single logarithm in the form $\log k$:

$$\log(8) + 1 = \boxed{\log(80)}$$

Answer:
$$\begin{aligned}\log(8) + 1 &= \log(8) + \log(10)\\ &= \log(8 \times 10)\\ &= \log(80)\end{aligned}$$

Ex 21: Write as a single logarithm in the form $\log k$:

$$\log(3) + 2 = \boxed{\log(300)}$$

Answer:
$$\begin{aligned}\log(3) + 2 &= \log(3) + \log(10^2)\\ &= \log(3) + \log(100)\\ &= \log(3 \times 100)\\ &= \log(300)\end{aligned}$$

Ex 22: Write as a single logarithm in the form $\log k$:

$$2 - \log(25) = \boxed{\log(4)}$$

Answer:
$$\begin{aligned}2 - \log(25) &= \log(10^2) - \log(25)\\ &= \log\left(\frac{100}{25}\right)\\ &= \log(4)\end{aligned}$$

Ex 23: Write as a single logarithm in the form $\log k$:

$$\log(200) - 2 = \boxed{\log(2)}$$

Answer:
$$\begin{aligned}\log(200) - 2 &= \log(200) - \log(10^2)\\ &= \log\left(\frac{200}{100}\right)\\ &= \log(2)\end{aligned}$$

B.3 WRITING AS A SINGLE LOGARITHM: LEVEL 3

Ex 24: Write as a single logarithm in the form $\log k$:

$$2\log(3) + 1 = \boxed{\log(90)}$$

Answer:
$$\begin{aligned}2\log(3) + 1 &= \log(3^2) + \log(10^1)\\ &= \log(9) + \log(10)\\ &= \log(9 \times 10)\\ &= \log(90)\end{aligned}$$

Ex 25: Write as a single logarithm in the form $\log k$:

$$3\log(2) - \log(4) = \boxed{\log(2)}$$

Answer:
$$\begin{aligned}3\log(2) - \log(4) &= \log(2^3) - \log(4)\\ &= \log(8) - \log(4)\\ &= \log\left(\frac{8}{4}\right)\\ &= \log(2)\end{aligned}$$

Ex 26: Write as a single logarithm in the form $\log k$:

$$2\log(20) - 2 = \boxed{\log(4)}$$

Answer:
$$\begin{aligned} 2\log(20) - 2 &= \log(20^2) - \log(10^2) \\ &= \log(400) - \log(100) \\ &= \log\left(\frac{400}{100}\right) \\ &= \log(4) \end{aligned}$$

Ex 27: Write as a single logarithm in the form $\log k$:

$$2\log(30) - 1 = \boxed{\log(90)}$$

Answer:
$$\begin{aligned} 2\log(30) - 1 &= \log(30^2) - \log(10^1) \\ &= \log(900) - \log(10) \\ &= \log\left(\frac{900}{10}\right) \\ &= \log(90) \end{aligned}$$

C USING LOGARITHMS TO SOLVE EXPONENTIAL EQUATIONS

C.1 SOLVING EXPONENTIAL EQUATIONS: LEVEL 1

Ex 28: Solve $2^x = 7$ (give your answer to 3 decimal places).

$$x = \boxed{2.807}$$

Answer:

$$\begin{aligned} 2^x &= 7 \\ \log(2^x) &= \log 7 && \text{(taking log of both sides)} \\ x\log 2 &= \log 7 && \text{(power rule)} \\ x &= \frac{\log 7}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 2.807 && \text{(using calculator)} \end{aligned}$$

Ex 29: Solve $3^x = 15$ (give your answer to 3 decimal places).

$$x = \boxed{2.465}$$

Answer:

$$\begin{aligned} 3^x &= 15 \\ \log(3^x) &= \log 15 && \text{(taking log of both sides)} \\ x\log 3 &= \log 15 && \text{(power rule)} \\ x &= \frac{\log 15}{\log 3} && \text{(dividing both sides by } \log 3\text{)} \\ x &\approx 2.465 && \text{(using calculator)} \end{aligned}$$

Ex 30: Solve $5^x = 100$ (give your answer to 3 decimal places).

$$x = \boxed{2.861}$$

Answer:

$$\begin{aligned} 5^x &= 100 \\ \log(5^x) &= \log 100 && \text{(taking log of both sides)} \\ x\log 5 &= \log 100 && \text{(power rule)} \\ x &= \frac{\log 100}{\log 5} && \text{(dividing both sides by } \log 5\text{)} \\ x &\approx 2.861 && \text{(using calculator)} \end{aligned}$$

Ex 31: Solve $6^x = 80$ (give your answer to 3 decimal places).

$$x = \boxed{2.446}$$

Answer:

$$\begin{aligned} 6^x &= 80 \\ \log(6^x) &= \log 80 && \text{(taking log of both sides)} \\ x\log 6 &= \log 80 && \text{(power rule)} \\ x &= \frac{\log 80}{\log 6} && \text{(dividing both sides by } \log 6\text{)} \\ x &\approx 2.446 && \text{(using calculator)} \end{aligned}$$

C.2 SOLVING EXPONENTIAL EQUATIONS: LEVEL 2

Ex 32: Solve $5 \cdot 2^x = 7$ (give your answer to 3 decimal places).

$$x = \boxed{0.485}$$

Answer:

$$\begin{aligned} 5 \cdot 2^x &= 7 \\ 2^x &= \frac{7}{5} && \text{(dividing both sides by 5)} \\ \log(2^x) &= \log\left(\frac{7}{5}\right) && \text{(taking log of both sides)} \\ x\log 2 &= \log\left(\frac{7}{5}\right) && \text{(power rule)} \\ x &= \frac{\log\left(\frac{7}{5}\right)}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 0.485 && \text{(using calculator)} \end{aligned}$$

Ex 33: Solve $-2^x = -10$ (give your answer to 3 decimal places).

$$x = \boxed{3.322}$$

Answer:

$$\begin{aligned} -2^x &= -10 \\ 2^x &= 10 && \text{(dividing both sides by } -1\text{)} \\ \log(2^x) &= \log 10 && \text{(taking log of both sides)} \\ x\log 2 &= \log 10 && \text{(power rule)} \\ x &= \frac{\log 10}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 3.322 && \text{(using calculator)} \end{aligned}$$

Ex 34: Solve $4 \cdot 3^x = 60$ (give your answer to 3 decimal places).

$$x = \boxed{2.465}$$

Answer:

$$\begin{aligned}
4 \cdot 3^x &= 60 \\
3^x &= \frac{60}{4} && \text{(dividing both sides by 4)} \\
3^x &= 15 \\
\log(3^x) &= \log 15 && \text{(taking log of both sides)} \\
x \log 3 &= \log 15 && \text{(power rule)} \\
x &= \frac{\log 15}{\log 3} && \text{(dividing both sides by } \log 3\text{)} \\
x &\approx 2.465 && \text{(using calculator)}
\end{aligned}$$

Ex 35: Solve $-2 \cdot (0.5)^x = -4$ (give your answer to 3 decimal places).

$$x = \boxed{-1.000}$$

Answer:

$$\begin{aligned}
-2 \cdot (0.5)^x &= -4 \\
(0.5)^x &= \frac{-4}{-2} && \text{(dividing both sides by } -2\text{)} \\
(0.5)^x &= 2 \\
\log\left((0.5)^x\right) &= \log(2) \\
x \cdot \log(0.5) &= \log(2) \\
x &= \frac{\log(2)}{\log(0.5)} \\
x &= -1
\end{aligned}$$

So, $x = -1$.

D APPLICATIONS OF LOGARITHMS

D.1 APPLYING OF LOGARITHMS IN SCIENCE

Ex 36: The pH scale in chemistry is $\text{pH} = -\log_{10}[H^+]$ where $[H^+]$ is the hydrogen ion concentration in moles per litre. The pH of a solution is 3.2. Find the hydrogen ion concentration $[H^+]$ (give your answer in scientific notation with 3 significant digits).

$$\boxed{6.31} \times \boxed{10^{-4}} \text{ mol/L}$$

Answer: We know:

$$\begin{aligned}
\text{pH} &= -\log_{10}[H^+] \\
3.2 &= -\log_{10}[H^+] && \text{(substituting the value)} \\
-3.2 &= \log_{10}[H^+] && \text{(multiplying both sides by } -1\text{)} \\
10^{-3.2} &= 10^{\log_{10}[H^+]} && \text{(exponentiating both sides)} \\
10^{-3.2} &= [H^+] && (10^{\log_{10} x} = x) \\
[H^+] &\approx 0.00063096 && \text{(using calculator)} \\
[H^+] &\approx 6.31 \times 10^{-4} \text{ mol/L} && \text{(in scientific notation with 3 significant digits)}
\end{aligned}$$

Ex 37: The Richter scale measures earthquake intensity using the formula $M = \log_{10}\left(\frac{I}{I_0}\right)$, where M is the magnitude, I is the intensity of the earthquake, and I_0 is the intensity of a standard earthquake.
An earthquake has a magnitude of 4.5 on the Richter scale. Find the intensity ratio $\frac{I}{I_0}$ (give your answer in scientific notation with 3 significant digits).

$$\frac{I}{I_0} = \boxed{3.16} \times \boxed{10^4}$$

Answer: We know:

$$\begin{aligned}
M &= \log_{10}\left(\frac{I}{I_0}\right) \\
4.5 &= \log_{10}\left(\frac{I}{I_0}\right) && \text{(substituting the value)} \\
10^{4.5} &= 10^{\log_{10}\left(\frac{I}{I_0}\right)} && \text{(exponentiating both sides)} \\
10^{4.5} &= \frac{I}{I_0} && (10^{\log_{10} x} = x) \\
\frac{I}{I_0} &\approx 31622.7766 && \text{(using calculator)} \\
\frac{I}{I_0} &\approx 3.16 \times 10^4 && \text{(in scientific notation with 3 significant digits)}
\end{aligned}$$

Ex 38: The intensity of sound is measured in decibels (dB) using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is the sound level in decibels, I is the intensity of the sound, and I_0 is the reference intensity (threshold of human hearing).
A sound has a level of 75 decibels. Find the intensity ratio $\frac{I}{I_0}$ (give your answer in scientific notation with 3 significant digits).

$$\frac{I}{I_0} = \boxed{3.16} \times \boxed{10^7}$$

Answer: We know:

$$\begin{aligned}
L &= 10\log_{10}\left(\frac{I}{I_0}\right) \\
75 &= 10\log_{10}\left(\frac{I}{I_0}\right) && \text{(substituting the value)} \\
7.5 &= \log_{10}\left(\frac{I}{I_0}\right) && \text{(dividing both sides by 10)} \\
10^{7.5} &= 10^{\log_{10}\left(\frac{I}{I_0}\right)} && \text{(exponentiating both sides)} \\
10^{7.5} &= \frac{I}{I_0} && (10^{\log_{10} x} = x) \\
\frac{I}{I_0} &\approx 3162277.66 && \text{(using calculator)} \\
\frac{I}{I_0} &\approx 3.16 \times 10^7 && \text{(in scientific notation with 3 significant digits)}
\end{aligned}$$

E GRAPHS OF LOGARITHMIC FUNCTIONS

E.1 FINDING DOMAINS

MCQ 39: Find the domain of the function $f : x \mapsto \log(x-4)$.

☐ $\mathbb{R}$

☐ $[-4, +\infty)$

☒ $(4, +\infty)$

☐ $(-\infty, 4)$

Answer: The function $f(x) = \log(x-4)$ is defined only when the argument of the logarithm is positive, i.e., when $x - 4 > 0$. Solving this inequality:

$$\begin{aligned} x - 4 &> 0 \\ x &> 4 \quad \text{(adding 4 to both sides)} \end{aligned}$$

Therefore, the function is defined for $x > 4$, so the domain is $(4, +\infty)$.

MCQ 40: Find the domain of the function $f : x \mapsto \log(2 - x)$.

☐ $\mathbb{R}$

☐ $[-2, +\infty)$

☐ $(2, +\infty)$

☒ $(-\infty, 2)$

Answer: The function $f(x) = \log(2 - x)$ is defined only when the argument of the logarithm is positive, i.e., when $2 - x > 0$. Solving this inequality:

$$\begin{aligned} 2 - x &> 0 \\ -x &> -2 \quad \text{(subtracting 2 from both sides)} \\ x &< 2 \quad \text{(multiplying both sides by } -1\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 2$, so the domain is $(-\infty, 2)$.

MCQ 41: Find the domain of the function $f : x \mapsto \log(2x - 6)$.

☐ $\mathbb{R}$

☐ $[3, +\infty)$

☒ $(3, +\infty)$

☐ $(-\infty, 3)$

Answer: The function $f(x) = \log(2x - 6)$ is defined only when the argument of the logarithm is positive, i.e., when $2x - 6 > 0$. Solving this inequality:

$$\begin{aligned} 2x - 6 &> 0 \\ 2x &> 6 \quad \text{(adding 6 to both sides)} \\ x &> 3 \quad \text{(dividing both sides by 2)} \end{aligned}$$

Therefore, the function is defined for $x > 3$, so the domain is $(3, +\infty)$. The correct answer is option (c).

MCQ 42: Find the domain of the function $f : x \mapsto \log(9 - 3x)$.

☐ $\mathbb{R}$

☐ $[3, +\infty)$

☐ $(3, +\infty)$

☒ $(-\infty, 3)$

Answer: The function $f(x) = \log(9 - 3x)$ is defined only when the argument of the logarithm is positive, i.e., when $9 - 3x > 0$. Solving this inequality:

$$\begin{aligned} 9 - 3x &> 0 \\ -3x &> -9 \quad \text{(subtracting 9 from both sides)} \\ x &< 3 \quad \text{(dividing both sides by } -3\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 3$, so the domain is $(-\infty, 3)$.

E.2 CALCULATING $f(x)$

Ex 43: For $f : x \mapsto 3\log(x)$, find in simplest form:

1. $f(1) = \boxed{0}$
2. $f(10) = \boxed{3}$

Answer:

1. $$\begin{aligned} f(1) &= 3\log(1) \\ &= 3 \cdot 0 \quad (\text{since } \log 1 = 0) \\ &= 0 \end{aligned}$$
2. $$\begin{aligned} f(10) &= 3\log(10) \\ &= 3 \cdot 1 \quad (\text{since } \log 10 = 1) \\ &= 3 \end{aligned}$$

Ex 44: For $f : x \mapsto \frac{1}{1+\log(x)}$, find in simplest form:

1. $f(1) = \boxed{1}$
2. $f(10) = \boxed{\frac{1}{2}}$

Answer:

1. $$\begin{aligned} f(1) &= \frac{1}{1+\log(1)} \\ &= \frac{1}{1+0} \quad (\text{since } \log 1 = 0) \\ &= 1 \end{aligned}$$
2. $$\begin{aligned} f(10) &= \frac{1}{1+\log(10)} \\ &= \frac{1}{1+1} \quad (\text{since } \log 10 = 1) \\ &= \frac{1}{2} \end{aligned}$$

Ex 45: For $f : x \mapsto x\log(x+1)$, find in simplest form:

1. $f(0) = \boxed{0}$
2. $f(1) = \boxed{\log(2)}$

Answer:

1. $$\begin{aligned} f(0) &= 0\log(0+1) \\ &= 0 \cdot \log(1) \\ &= 0 \cdot 0 \\ &= 0 \end{aligned}$$
2. $$\begin{aligned} f(1) &= 1\log(1+1) \\ &= 1 \cdot \log(2) \\ &= \log(2) \end{aligned}$$