LOGARITHMS

In previous chapters, we explored exponential functions of the form $f(x) = 10^x$. This chapter introduces the inverse of these functions: logarithms. Logarithms are essential for solving equations where the unknown is in the exponent, and they have wide applications in science, engineering, and finance, such as measuring pH levels, earthquake magnitudes, and sound intensity.

A DEFINITION

Definition Logarithm

The **logarithm** of a number y (where $y > 0$) is the exponent to which 10 must be raised to obtain y. It is denoted as:

$$\log y = x \quad \Leftrightarrow \quad 10^x = y$$

Proposition Inverse Properties of Logarithms

For any $x > 0$:

- $10^{\log x} = x$
- $\log(10^x) = x$

These properties reflect that the exponential function 10^x and the logarithm function $\log x$ are inverses of each other.

Ex: Evaluate $\log 100$.

Answer:
$$\begin{aligned} \log(100) &= \log\left(10^2\right) \\ &= 2 \end{aligned}$$

B LAWS OF LOGARITHMS

Proposition Laws of Logarithms

For positive real numbers x, y:

- Product Rule: $\log(xy) = \log x + \log y$
- Quotient Rule: $\log\left(\frac{x}{y}\right) = \log x - \log y$
- Power Rule: $\log(x^k) = k \log x$

Ex: Write as a single logarithm $\log(5) + \log(3)$.

Answer:
$$\begin{aligned} \log(5) + \log(3) &= \log(5 \times 3) \\ &= \log 15 \end{aligned}$$

C USING LOGARITHMS TO SOLVE EXPONENTIAL EQUATIONS

Method Solving Exponential Equations Using Logarithms

To solve an exponential equation of the form $a^x = b$ (where $a > 0$, $a \neq 1$, $b > 0$):

1. Take the logarithm (base 10) of both sides: $\log(a^x) = \log b$
2. Apply the power rule: $x \log a = \log b$
3. Solve for x: $x = \frac{\log b}{\log a}$

Ex: Solve $2^x = 7$.

Answer:
$$\begin{aligned} 2^x &= 7 \\ \log(2^x) &= \log 7 && \text{(taking log of both sides)} \\ x \log 2 &= \log 7 && \text{(power rule)} \\ x &= \frac{\log 7}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 2.807 && \text{(using calculator)} \end{aligned}$$

D APPLICATIONS OF LOGARITHMS

Logarithms are used to describe quantities that vary over many orders of magnitude. Some important examples in science include:

- **pH scale** in chemistry: $\text{pH} = -\log_{10}[H^+]$
  (where $[H^+]$ is the hydrogen ion concentration in moles per litre)
- **Richter scale** for earthquakes: Magnitude $= \log_{10}\left(\frac{I}{I_0}\right)$
  (where I is the intensity of the earthquake, I_0 is a reference intensity)
- **Decibel (dB) scale** for sound: $L = 10\log_{10}\left(\frac{P}{P_0}\right)$
  (where P is the power/intensity measured, P_0 is a reference level)

Ex: The pH of a solution is 3.2. Find the hydrogen ion concentration $[H^+]$.

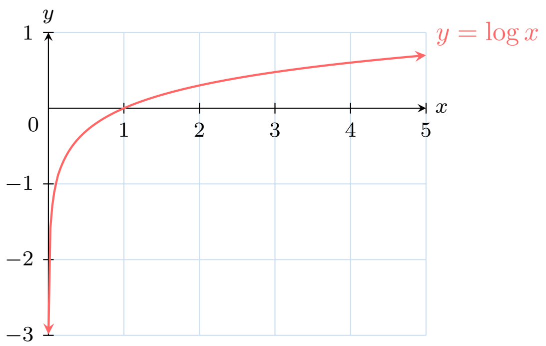
Answer: We know:

$$\begin{aligned}
\text{pH} &= -\log_{10}[H^+] \\
3.2 &= -\log_{10}[H^+] && \text{(substituting the value)} \\
-3.2 &= \log_{10}[H^+] && \text{(multiplying both sides by } -1) \\
10^{-3.2} &= 10^{\log_{10}[H^+]} && \text{(exponentiating both sides)} \\
[H^+] &= 10^{-3.2} \\
[H^+] &\approx 6.31 \times 10^{-4} \text{ mol/L} && \text{(using calculator)}
\end{aligned}$$

E GRAPHS OF LOGARITHMIC FUNCTIONS

Definition Logarithmic Function

The **logarithmic function** is $x \mapsto \log x$ with domain $(0, +\infty)$.



Proposition Graph Properties

The graph of $y = \log x$ is the inverse of $y = 10^x$. It has a vertical asymptote at $x = 0$, passes through (1,0), and is increasing.