MACLAURIN SERIES

A MACLAURIN SERIES

A.1 EXPANDING MACLAURIN SERIES FROM SIGMA NOTATION

Ex 1: Expand the series for $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ up to the term of order 3.

$$e^x = \boxed{\quad\quad\quad\quad} + \sum_{k=4}^{\infty} \frac{x^k}{k!}$$

Ex 2: Expand the series for $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ up to the term of order 3.

$$\frac{1}{1+x} = \boxed{\quad\quad\quad\quad} + \sum_{k=4}^{\infty} (-1)^k x^k$$

Ex 3: Expand the series for $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$ up to the term of order 4.

$$\ln(1+x) = \boxed{\quad\quad\quad\quad} + \sum_{k=5}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

Ex 4: Expand the series for $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ up to the term of order 5.

$$\sin(x) = \boxed{\quad\quad\quad\quad} + \sum_{k=3}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

A.2 WRITING A SERIES USING SIGMA NOTATION

Ex 5: For the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$, write the series using sigma notation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \boxed{\quad}$$

Ex 6: For the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, write the series using sigma notation.

$$e^x = \sum_{n=0}^{\infty} \boxed{\quad}$$

Ex 7: For the series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, write the series using sigma notation.

$$\ln(1+x) = \sum_{n=1}^{\infty} \boxed{\quad\quad\quad}$$

Ex 8: For the series $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$, write the series using sigma notation.

$$\cos(x) = \sum_{n=0}^{\infty} \boxed{\quad\quad\quad}$$

A.3 DERIVING STANDARD MACLAURIN SERIES FROM FIRST PRINCIPLES

Ex 9: For the function $f(x) = e^x$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.
3. Show that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Ex 10: For the function $f(x) = \cos(x)$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$.
3. Show that the Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Ex 11: For the function $f(x) = \frac{1}{1-x}$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.
3. Show that the Maclaurin series for $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

Ex 12: For the function $f(x) = \ln(1-x)$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.
3. Show that the Maclaurin series for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

A.4 FINDING SERIES WITH THE BINOMIAL FORMULA

Ex 13: The general Maclaurin series for the function $f(x) = (1+x)^p$, known as the binomial series, is given by:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

Use this formula to find the first four non-zero terms of the Maclaurin series for $f(x) = \sqrt{1+x}$.

$$\sqrt{1+x} = \boxed{} + \ldots$$

Ex 14: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \frac{1}{(1+x)^2}$.

$$\frac{1}{(1+x)^2} = \boxed{} + \ldots$$

Ex 15: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

to find the Maclaurin series for $f(x) = (1+x)^3$. Explain why the series terminates.

$$(1+x)^3 = \boxed{}$$

B MACLAURIN POLYNOMIALS FOR APPROXIMATION

B.1 FINDING MACLAURIN POLYNOMIALS FROM A GIVEN SERIES

Ex 16: Given the Maclaurin series for $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$, write down the Maclaurin polynomial of degree 3, $P_3(x)$.

$$P_3(x) = \boxed{}$$

Ex 17: Given the Maclaurin series $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$, write down the Maclaurin polynomial of degree 4, $P_4(x)$, in expanded form.

$$P_4(x) = \boxed{}$$

Ex 18: Given the Maclaurin series for $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$, write down the Maclaurin polynomial of degree 4, $P_4(x)$.

$$P_4(x) = \boxed{}$$

B.2 FINDING MACLAURIN POLYNOMIALS FROM FIRST PRINCIPLES

Ex 19: Find the Maclaurin polynomial of degree 3 for the function $f(x) = e^{2x}$.

$$P_3(x) = \boxed{}$$

Ex 20: Find the Maclaurin polynomial of degree 3 for the function $f(x) = \ln(1+x)$.

$$P_3(x) = \boxed{}$$

Ex 21: Find the Maclaurin polynomial of degree 2 for the function $f(x) = e^{x^2}$.

$$P_2(x) = \boxed{}$$

B.3 APPROXIMATING FUNCTION VALUES USING MACLAURIN POLYNOMIALS

Ex 22: The Maclaurin polynomial of degree 3 for the function $f(x) = \ln(1+x)$ is

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Use this polynomial to find an approximation for $\ln(1.4)$ (round to 3 decimal places).

$$\ln(1.4) \approx \boxed{\quad\quad}$$

Ex 23: The Maclaurin polynomial of degree 3 for the function $f(x) = e^x$ is

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Use this polynomial to find an approximation for e (round to 3 decimal places).

$$e \approx \boxed{\quad\quad}$$

Ex 24: The Maclaurin polynomial of degree 5 for the function $f(x) = \arctan(x)$ is

$$P_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Use this polynomial and the fact that $\arctan(1) = \frac{\pi}{4}$ to find an approximation for π (round to 3 decimal places).

$$\pi \approx \boxed{\quad\quad}$$

B.4 ESTIMATING THE ERROR OF MACLAURIN APPROXIMATIONS

Ex 25: Consider the function $f(x) = \cos(x)$.

1. Find the Maclaurin polynomial of degree 4, $P_4(x)$, for $f(x) = \cos(x)$.
2. Use this polynomial to approximate the value of $\cos(0.5)$.
3. The Lagrange form of the remainder term, $R_n(x)$, gives the exact error of a Maclaurin approximation of degree n, and is defined as:

   $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

   for some value c between 0 and x. Write down the Lagrange form of the remainder $R_4(x)$ for the approximation in part (b).
4. By finding the maximum possible absolute value of $R_4(0.5)$, determine the upper bound for the error in your approximation of $\cos(0.5)$.

Ex 26: Consider the function $f(x) = \ln(1+x)$.

1. Find the Maclaurin polynomial of degree 3, $P_3(x)$, for $f(x) = \ln(1+x)$.
2. Use this polynomial to approximate the value of $\ln(1.2)$.
3. The Lagrange form of the remainder term, $R_n(x)$, is given by:

   $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

   for some value c between 0 and x. Write down the Lagrange form of the remainder $R_3(x)$ for the approximation in part (b).
4. By finding the maximum possible absolute value of $R_3(0.2)$, determine the upper bound for the error in your approximation of $\ln(1.2)$.

C SUBSTITUTION AND DIFFERENTIATION/INTEGRATION TERM-BY-TERM

C.1 FINDING NEW SERIES BY SUBSTITUTION

Ex 27: Starting with the geometric series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$, find the Maclaurin series for $f(x) = \frac{1}{1-2x}$.

Ex 28: Starting with the Maclaurin series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$, find the Maclaurin series for $f(x) = e^{x^2}$.

Ex 29: Starting with the Maclaurin series $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$, find the Maclaurin series for $f(x) = \cos(2x)$.

Ex 30: Starting with the Maclaurin series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$, use a substitution to find the first four non-zero terms of the series for $f(x) = e^{ix}$.

C.2 FINDING NEW SERIES WITH THE BINOMIAL FORMULA BY SUBSTITUTION

Ex 31: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \frac{1}{1-2x}$.

$$\frac{1}{1-2x} = \boxed{\quad} + \dots$$

Ex 32: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \sqrt{1+x^2}$.

$$\sqrt{1+x^2} = \boxed{\quad} + \dots$$

Ex 33: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots$$

to find the first three non-zero terms of the Maclaurin series for $f(x) = \sqrt{2+x}$.

$$\sqrt{2+x} = \boxed{\quad} + \dots$$

C.3 DIFFERENTIATING MACLAURIN POLYNOMIALS

Ex 34: Find the derivative of the Maclaurin polynomial of degree 4 for $\frac{1}{1-x}$: $P_4(x) = 1 + x + x^2 + x^3 + x^4$.

$$P_4'(x) = \boxed{}$$

Ex 35: Find the derivative of the Maclaurin polynomial of degree 5 for e^x:

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$P_5'(x) = \boxed{}$$

Ex 36: Find the derivative of the Maclaurin polynomial of degree 5 for $\ln(1+x)$:

$$P_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$P_5'(x) = \boxed{}$$

Ex 37: Find the derivative of the Maclaurin polynomial of degree 4 for $\cos(x)$:

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_4'(x) = \boxed{}$$

C.4 INTEGRATING MACLAURIN POLYNOMIALS

Ex 38: Find the indefinite integral of the Maclaurin polynomial of degree 3 for $\frac{1}{1-x}$: $P_3(x) = 1 + x + x^2 + x^3$.

$$\int P_3(x)dx = \boxed{}$$

Ex 39: Find the indefinite integral of the Maclaurin polynomial of degree 4 for e^x:

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\int P_4(x)dx = \boxed{}$$

Ex 40: Find the indefinite integral of the Maclaurin polynomial of degree 3 for $\cos(x)$:

$$P_3(x) = 1 - \frac{x^2}{2!}$$

$$\int P_3(x)dx = \boxed{}$$

C.5 FINDING NEW SERIES BY DIFFERENTIATION

Ex 41: Consider the Maclaurin series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

By differentiating both sides of this equation, find the Maclaurin series for the function $\frac{1}{(1-x)^2}$.

Ex 42: Consider the Maclaurin series for $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

By differentiating both sides of this equation, find the Maclaurin series for the function $f(x) = \frac{1}{1+x}$.

Ex 43: Consider the Maclaurin series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

By differentiating both sides of this equation, find the Maclaurin series for the function $f(x) = \cos(x)$.

Ex 44: Consider the Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Show that differentiating the series term-by-term reproduces the original series.

C.6 FINDING NEW SERIES BY INTEGRATION

Ex 45: Consider the Maclaurin series:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$

By integrating both sides of this equation from 0 to x, find the Maclaurin series for the function $f(x) = \ln(1+x)$.

Ex 46: Consider the Maclaurin series for $\cos(t)$:

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$

By integrating both sides of this equation from 0 to x, find the Maclaurin series for the function $f(x) = \sin(x)$.

Ex 47:

1. Starting with the geometric series $\frac{1}{1-u} = 1 + u + u^2 + \dots$, use a substitution to find the Maclaurin series for $\frac{1}{1+t^2}$.
2. By integrating the resulting series from 0 to x, find the Maclaurin series for $f(x) = \arctan(x)$.

D LINEARITY OF MACLAURIN SERIES

D.1 COMBINING SERIES TO FIND NEW SERIES

Ex 48: This exercise guides you to find the Maclaurin series for the hyperbolic cosine function, $\cosh(x)$.

1. Start with the Maclaurin series for e^u. By substituting $u = -x$, find the Maclaurin series for $f(x) = e^{-x}$.

2. The hyperbolic cosine is defined as $\cosh(x) = \frac{e^x+e^{-x}}{2}$. Use your series for e^x and e^{-x} to find the Maclaurin series for $\cosh(x)$.

Ex 49: This exercise guides you to find the Maclaurin series for the hyperbolic sine function, $\sinh(x)$.

1. You have already found the series for e^x and e^{-x}. Recall them here.

2. The hyperbolic sine is defined as $\sinh(x) = \frac{e^x-e^{-x}}{2}$. Use your series for e^x and e^{-x} to find the Maclaurin series for $\sinh(x)$.

Ex 50: Consider the Maclaurin series for the real exponential function:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \dots$$

By formally substituting $u = ix$ and rearranging the terms, show how this series relates to the series for $\cos(x)$ and $\sin(x)$.