PROPERTIES OF INTEGERS

A NUMBERS 1 AND 0

Zero (0) and one (1) are very special numbers that have important properties.

Proposition Additive Identity Property

Adding 0 to any number results in the number itself.
For any number a,

$$0 + a = a \quad \text{and} \quad a + 0 = a.$$

Ex: $7 + 0 = 7$

Proposition Multiplicative Identity Property

Multiplying any number by 1 results in the number itself.
For any number a,

$$1 \times a = a \quad \text{and} \quad a \times 1 = a.$$

Ex: $1 \times 7 = 7$

Proposition Multiplying by 0

For any number a,

$$a \times 0 = 0 \quad \text{and} \quad 0 \times a = 0.$$

Proposition Dividing by 0

For any number a, the division $a \div 0$ is **undefined**.

B DIVISION WITH REMAINDERS

Theorem Division with Remainder

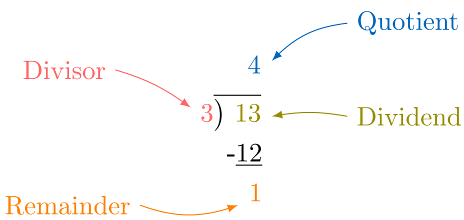
For any natural integer a and any non-zero natural integer b, there exist unique natural integers q and r such that

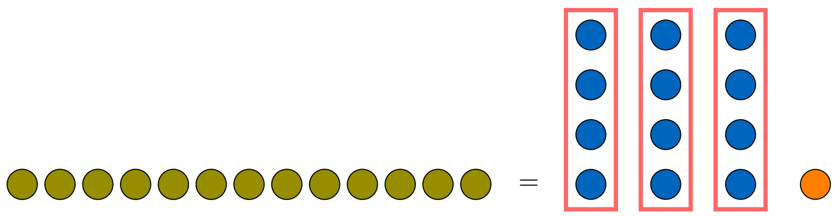
$$a = b \times q + r \quad \text{with} \quad 0 \leq r < b.$$

- a is the dividend (the number to be shared),
- b is the divisor (the number we divide by / the number of groups),
- q is the quotient (the size of each group),
- r is the remainder.

Ex:

$$\begin{array}{ccccccccc} 13 & = & 3 & \times & 4 & + & 1 & \text{with} & 0 \leq 1 < 3 \\ \text{Dividend} & = & \text{Divisor} & \times & \text{Quotient} & + & \text{Remainder} & \text{with} & 0 \leq \text{Remainder} < \text{Divisor} \end{array}$$





C DIVISIBILITY

Definition Divisibility Relationships

We say that a non-zero natural integer b **divides** a natural integer a if a can be obtained by multiplying b by another integer k:

$$a = k \times b$$

In other words, the number a appears in the multiplication table of b.
We can also use the following formulations:

- b is a **divisor** of a
- b is a **factor** of a
- a is **divisible** by b
- a is a **multiple** of b

Ex: Consider the numbers 10 and 5.
Since we can write $10 = 2 \times 5$, we can say that:

- 5 divides 10.
- 5 is a divisor (or factor) of 10.
- 10 is divisible by 5.
- 10 is a multiple of 5.

Method Check divisibility

To check if a number a is divisible by a number b, perform the division with Remainder of a by b.

- If the **remainder is zero**, then a is divisible by b.
- If the **remainder is not zero**, then a is not divisible by b.

Ex: Is 13 divisible by 5?

Answer: We perform the division of 13 by 5:

$$13 = 5 \times 2 + 3$$

The remainder is 3 (which is not zero). Therefore, 13 is **not** divisible by 5.

D DIVISIBILITY CRITERIA

Divisibility criteria are methods that allow us to quickly determine whether a whole number is divisible by another whole number without performing long division. These rules are useful for simplifying calculations and for understanding properties of numbers. Here are some common divisibility criteria:

Proposition Divisibility Criteria for 2 and 5

- A number is divisible by 2 if its last digit is even $(0, 2, 4, 6 \text{ or } 8)$.
- A number is divisible by 5 if its last digit is 0 or 5.

Ex: Determine whether 946 is divisible by 2.

Answer: 946 is divisible by 2 because its last digit is 6, which is even.

Ex: Determine whether 947 is divisible by 5.

Answer: 947 is not divisible by 5 because its last digit is 7, which is not 0 or 5.

Proposition Divisibility Criteria for 3 and 9

- A number is divisible by 3 if the sum of its digits is divisible by 3.
- A number is divisible by 9 if the sum of its digits is divisible by 9.

Ex: Determine whether 948 is divisible by 3.

Answer: 948 is divisible by 3 because the sum of its digits, $9+4+8=21$, is divisible by 3 ($21=3\times 7$).

Ex: Determine whether 948 is divisible by 9.

Answer: 948 is not divisible by 9 because the sum of its digits, $9+4+8=21$, is not divisible by 9 ($21=9\times 2+3$).

Proposition **Divisibility Criteria for 4**

A number is divisible by 4 if the number formed by its last two digits is divisible by 4.

Ex: Determine whether 917 is divisible by 4.

Answer: 917 is not divisible by 4 because the number formed by its last two digits, 17, is not divisible by 4 ($17=4\times 4+1$).

E PRIME NUMBER

Definition **Prime Number**

A **prime number** is a whole number greater than 1 that has **only two different divisors**: 1 and itself.
A whole number greater than 1 that is not prime is called a **composite number**.

Ex: State whether 6 is a prime number.

Answer: As $2\times 3=6$, 6 is not a prime number because it is divisible by 2 and 3, in addition to 1 and 6. So 6 is a composite number.

Ex: State whether 5 is a prime number.

Answer: 5 is a prime number because it is divisible (without remainder) only by 1 and 5. We cannot divide it exactly by 2, 3 or 4 (there would be a remainder).

Proposition **First 25 Prime Numbers**

Here is the list of the first 25 prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

F PRIME FACTORIZATION

Method **Prime Factorization**

Prime factorization of a number is writing that number as a product of **prime numbers only**. In other words, it is finding which prime numbers you need to multiply together to get the original number.

Ex: Find the prime factorization of 12.

Answer: The prime factorization is $12=2\times 2\times 3$.
The order is not important. You can write $12=3\times 2\times 2$.
The prime factorization is not $12=2\times 6$ because 6 is a composite number.

Method **Factor Tree**

The **factor tree method** involves breaking down a composite number into smaller factors, then breaking down those factors further until you have only prime factors.

1. Place the number at the top of the factor tree.
2. **Check if the number is prime.**
   (a) **If the number is prime:** Circle it. You are done with this branch.
   (b) **If the number is composite:** Factor it into two smaller factors. Write these two factors as branches below the number. Repeat step 2 for each of these new factors.
3. The prime factorization is the product of all circled prime numbers on the tree.

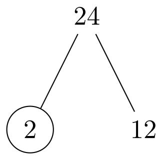
Ex: Find a prime factorization of 24.

Answer:

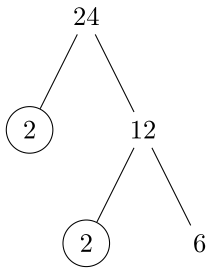
- Step 1:

24

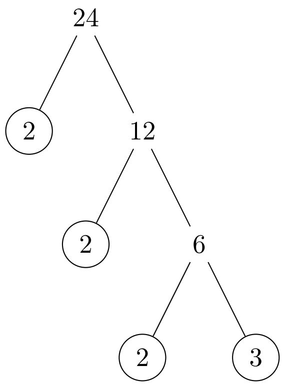
- Step 2: 24 is a composite number. $24 = 2 \times 12$.



- Step 3: 12 is a composite number. $12 = 2 \times 6$.



- Step 4: 6 is a composite number. $6 = 2 \times 3$.



A prime factorization is $24 = 2 \times 2 \times 2 \times 3$.

G GREATEST COMMON DIVISOR (GCD)

Definition Greatest Common Divisor (GCD)

The **greatest common divisor (GCD)** of two (or more) whole numbers is the largest positive integer that divides each of them exactly (with remainder 0). It is also called the **highest common factor (HCF)**.
We usually write the GCD of a and b as $\gcd(a, b)$.

Ex: Find the GCD of 18 and 24.

Answer:

- The divisors of 18 are: $1, 2, 3, 6, 9, 18$.
- The divisors of 24 are: $1, 2, 3, 4, 6, 8, 12, 24$.
- The greatest common divisor is 6, so $\gcd(18, 24) = 6$.

Method Finding GCD Using Prime Factorization

1. Write the prime factorization of each number.
2. Identify the prime factors that appear in **both** factorizations, taking each common prime as many times as it appears in *both* numbers (i.e. with the smallest exponent).
3. Multiply these common factors to get the GCD.

Ex: Find the GCD of 18 and 24 using prime factorization.

Answer:

- $18 = \boxed{2} \times \boxed{3} \times 3 = 2 \times 3^2$
- $24 = \boxed{2} \times 2 \times 2 \times \boxed{3} = 2^3 \times 3$
- Common prime factors (with smallest exponents): $\boxed{2}$ and $\boxed{3}$.
- So, $\gcd(18, 24) = 2 \times 3 = 6$.

H LEAST COMMON MULTIPLE (LCM)

Definition Least Common Multiple (LCM)

The **least common multiple (LCM)** of two (or more) whole numbers is the smallest positive integer that is a multiple of each of them. It is also called the **lowest common multiple**.
We usually write the LCM of a and b as $\text{lcm}(a, b)$.

Ex: Find the LCM of 8 and 12.

Answer:

- The multiples of 8 are: $8, 16, 24, 32, 40, \ldots$
- The multiples of 12 are: $12, 24, 36, 48, \ldots$
- The least common multiple is 24, so $\text{lcm}(8, 12) = 24$.

Method Finding LCM Using Prime Factorization

1. Write the prime factorization of each number.
2. For each prime factor, take the highest power that appears in either number.
3. Multiply these factors together to get the LCM.

Ex: Find the LCM of 8 and 12 using prime factorization.

Answer:

- $8 = 2 \times 2 \times 2 = 2^3$
- $12 = 2 \times 2 \times 3 = 2^2 \times 3^1$
- Take the highest powers of each prime: 2^3 and 3^1.
- So, $\text{lcm}(8, 12) = 2^3 \times 3^1 = 8 \times 3 = 24$.