QUADRATIC FUNCTIONS

A DEFINITION

A.1 RECOGNIZING QUADRATIC FUNCTIONS

MCQ 1: Is $f(x) = 2x^2 - 3x + 2$ a quadratic function?

☒ Yes.

☐ No.

Answer: $f(x) = 2x^2 - 3x + 2$ is a quadratic function because it has the form $ax^2 + bx + c$ with $a = 2$, $b = -3$, and $c = 2$, and contains a non-zero x^2 term.
The correct choice is: Yes.

MCQ 2: Is $f(x) = 2x - 3$ a quadratic function?

☐ Yes.

☒ No.

Answer: $f(x) = 2x - 3$ is not a quadratic function because it lacks an x^2 term; it is a linear function of the form $ax + b$.
The correct choice is: No.

MCQ 3: Is $f(x) = 2x^2 - 3x + \frac{1}{x}$ a quadratic function?

☐ Yes.

☒ No.

Answer: $f(x) = 2x^2 - 3x + \frac{1}{x}$ is not a quadratic function because it contains a term $\frac{1}{x}$, which makes it a rational function rather than a polynomial of degree 2.
The correct choice is: No.

MCQ 4: Is $f(x) = (x - 1)(x + 2)$ a quadratic function?

☒ Yes.

☐ No.

Answer: $f(x) = (x - 1)(x + 2)$ is a quadratic function because, when expanded, it becomes $f(x) = x^2 + x - 2$, which has the form $ax^2 + bx + c$ with $a = 1$, $b = 1$, and $c = -2$, and contains a non-zero x^2 term.
The correct choice is: Yes.

A.2 CALCULATING $f(x)$

Ex 5: For $f : x \mapsto x^2 - 3x + 1$,

- $f(0) = \boxed{1}$
- $f(2) = \boxed{-1}$
- $f\left(\frac{1}{2}\right) = \boxed{-\frac{1}{4}}$

Answer:

- $f(0) = 0^2 - 3(0) + 1$ (substituting x with 0)
  $= 0 - 0 + 1$
  $= 1$
- $f(2) = 2^2 - 3(2) + 1$ (substituting x with 2)
  $= 4 - 6 + 1$
  $= -1$
- $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 1$ (substituting x with $\frac{1}{2}$)
  $= \frac{1}{4} - \frac{3}{2} + 1$
  $= \frac{1}{4} - \frac{3}{2} + \frac{4}{4}$
  $= \frac{1}{4} - \frac{6}{4} + \frac{4}{4}$
  $= -\frac{1}{4}$

Ex 6: For $f : x \mapsto (x - 1)(x - 2)$,

- $f(0) = \boxed{2}$
- $f(2) = \boxed{0}$
- $f\left(\frac{1}{2}\right) = \boxed{\frac{3}{4}}$

Answer:

- $f(0) = (0 - 1)(0 - 2)$ (substituting x with 0)
  $= (-1) \times (-2)$
  $= 2$
- $f(2) = (2 - 1)(2 - 2)$ (substituting x with 2)
  $= (1) \times (0)$
  $= 0$
- $f\left(\frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)$ (substituting x with $\frac{1}{2}$)
  $= \left(\frac{1}{2} - \frac{2}{2}\right)\left(\frac{1}{2} - -\frac{4}{2}\right)$
  $= \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right)$
  $= \frac{3}{4}$

Ex 7: For $f : x \mapsto (x - 2)^2 + 4$,

- $f(2) = \boxed{4}$
- $f(1) = \boxed{5}$
- $f(3) = \boxed{5}$

Answer:

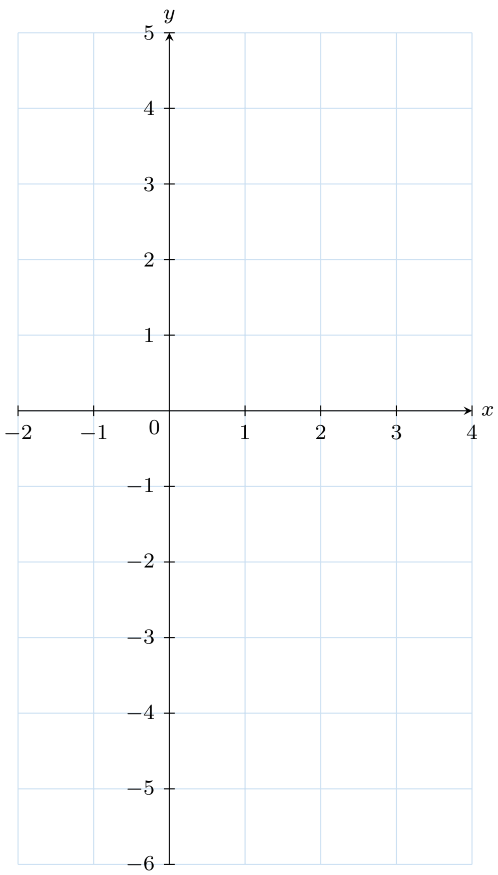
- $f(2) = (2 - 2)^2 + 4$ (substituting x with 2)
  $= 0^2 + 4$
  $= 4$
- $f(1) = (1 - 2)^2 + 4$ (substituting x with 1)
  $= (-1)^2 + 4$
  $= 1 + 4$
  $= 5$

- $f(3) = (3-2)^2 + 4 \quad$ (substituting x with 3)
  $= 1^2 + 4$
  $= 1 + 4$
  $= 5$

B GRAPH

B.1 PLOTTING GRAPHS

Ex 8: For the function $f(x) = -x^2+2x+3$, sketch the graph of f.(You may fill in a table of values for $x = -2, -1, 0, 1, 2, 3, 4$.)



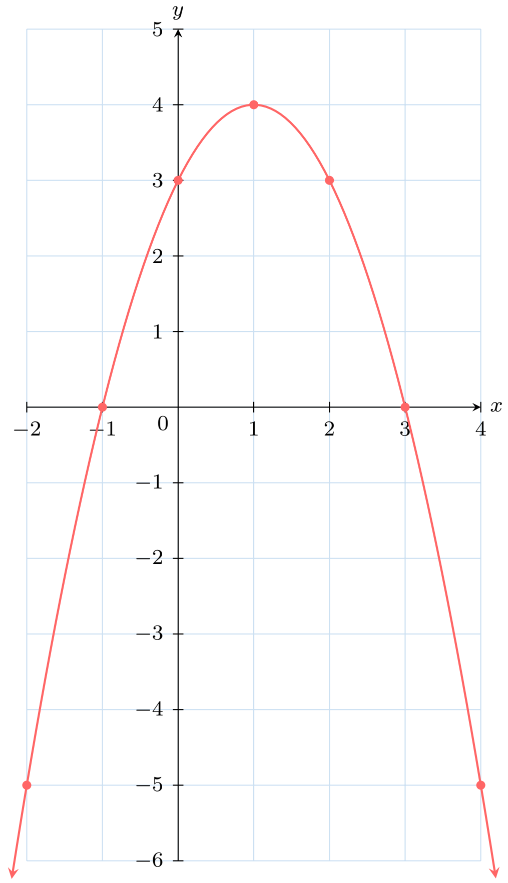
Answer: Step-by-step calculations:

$$f(-2) = -(-2)^2 + 2(-2) + 3 = -4 - 4 + 3 = -5$$
$$f(-1) = -(-1)^2 + 2(-1) + 3 = -1 - 2 + 3 = 0$$
$$f(0) = -0 + 0 + 3 = 3$$
$$f(1) = -1 + 2 + 3 = 4$$
$$f(2) = -4 + 4 + 3 = 3$$
$$f(3) = -9 + 6 + 3 = 0$$
$$f(4) = -16 + 8 + 3 = -5$$

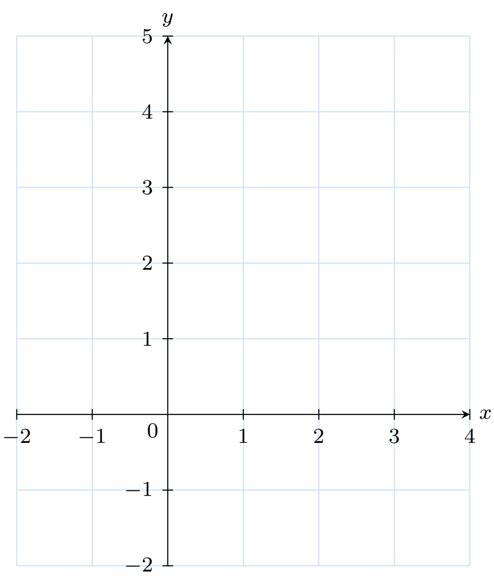
Fill in the table of values:

x	-2	-1	0	1	2	3	4
$f(x)$	-5	0	3	4	3	0	-5

Plot the points and draw the graph:



Ex 9: For the function $f(x) = \dfrac{x^2}{2} - 2x - 1$, sketch the graph of f.(You may fill in a table of values for $x = -2, -1, 0, 1, 2, 3, 4$.)



Answer: Step-by-step calculations:

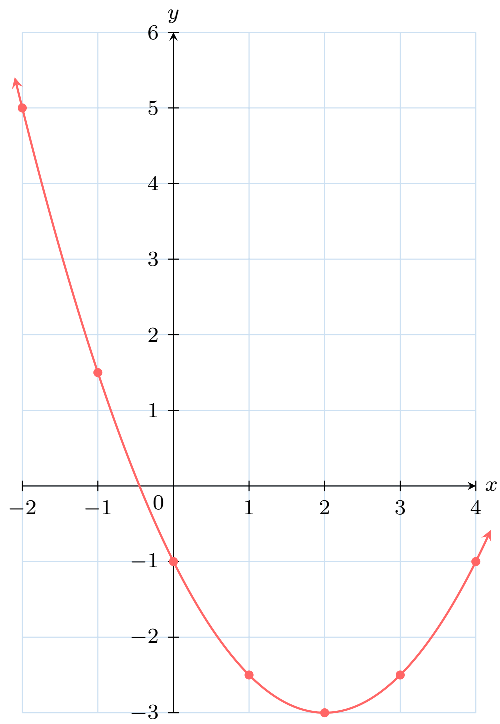
$$f(-2) = \frac{(-2)^2}{2} - 2 \times (-2) - 1 = \frac{4}{2} + 4 - 1 = 2 + 4 - 1 = 5$$
$$f(-1) = \frac{1}{2} - 2 \times (-1) - 1 = 0.5 + 2 - 1 = 1.5$$
$$f(0) = 0 - 0 - 1 = -1$$
$$f(1) = \frac{1}{2} - 2 - 1 = 0.5 - 2 - 1 = -2.5$$
$$f(2) = \frac{4}{2} - 4 - 1 = 2 - 4 - 1 = -3$$
$$f(3) = \frac{9}{2} - 6 - 1 = 4.5 - 6 - 1 = -2.5$$
$$f(4) = \frac{16}{2} - 8 - 1 = 8 - 8 - 1 = -1$$

Fill in the table of values:

x	-2	-1	0	1	2	3	4
$f(x)$	5	0.5	-1	-2.5	-3	-2.5	-1

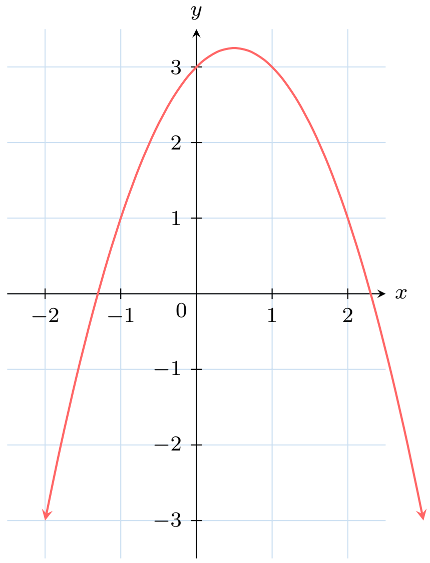
x	-2	-1	0	1	2	3	4
$f(x)$	5	1.5	-1	-2.5	-3	-2.5	-1

Plot the points and draw the graph:



B.2 DETERMINING THE SIGN OF a FROM THE GRAPH

MCQ 10: For the quadratic function $f(x) = ax^2 + bx + c$ with this following graph



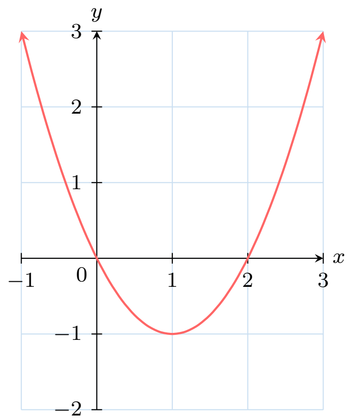
State the sign of a

☐ $a > 0$

☒ $a < 0$

Answer: The graph of the quadratic function opens downward (concave down), which indicates that the coefficient a is negative.The correct choice is: $a < 0$.

MCQ 11: For the quadratic function $f(x) = ax^2 + bx + c$ with the following graph



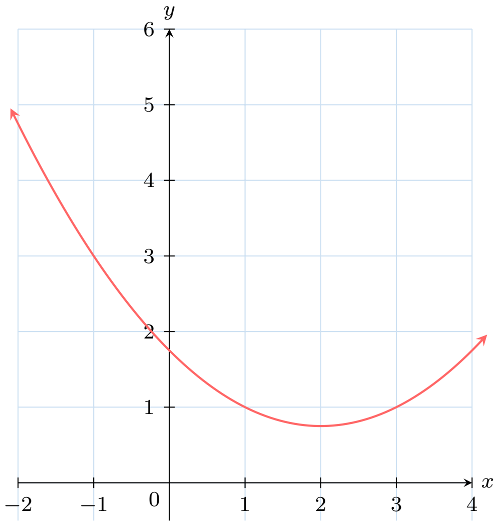
State the sign of a

☒ $a > 0$

☐ $a < 0$

Answer: The graph of the quadratic function opens upward (concave up), which indicates that the coefficient a is positive.The correct choice is: $a > 0$.

MCQ 12: For the quadratic function $f(x) = ax^2 + bx + c$ with the following graph



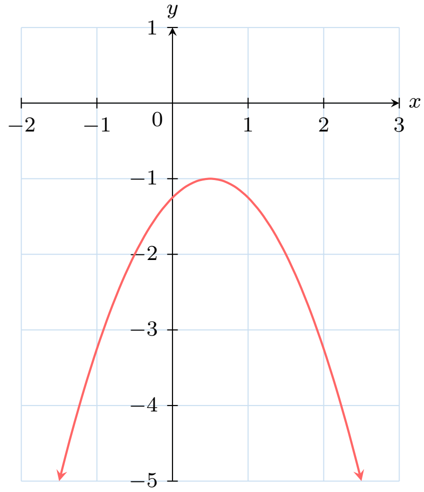
State the sign of a

☒ $a > 0$

☐ $a < 0$

Answer: The graph of the quadratic function opens upward (concave up), which indicates that the coefficient a is positive.The correct choice is: $a > 0$.

MCQ 13: For the quadratic function $f(x) = ax^2 + bx + c$ with the following graph



State the sign of a

☐ $a > 0$

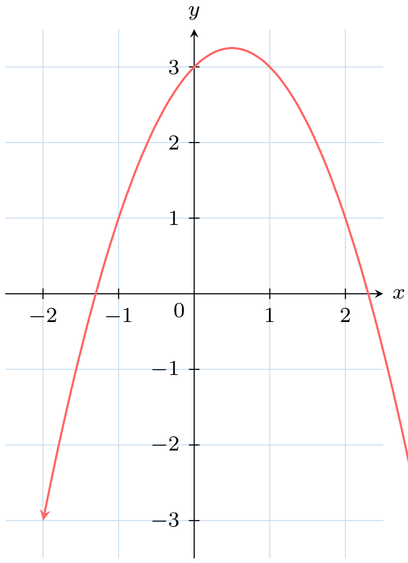
☒ $a < 0$

Answer: The graph of the quadratic function opens downward (concave down), which indicates that the coefficient a is negative.The correct choice is: $a < 0$.

C SOLVING $f(x) = y$

C.1 FINDING x SUCH THAT $f(x) = y$ GRAPHICALLY

Ex 14: The graph of $y = f(x)$ is:

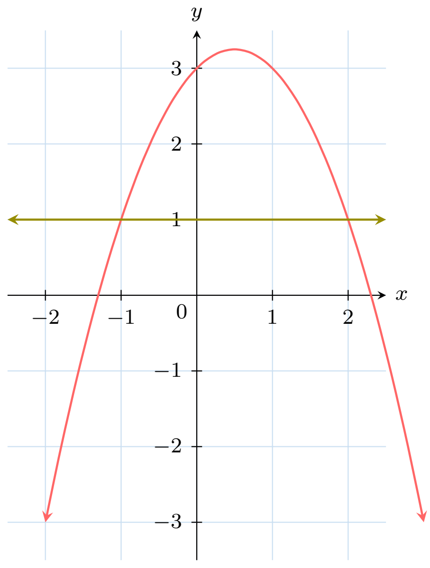


Find x such that $f(x) = 1$.

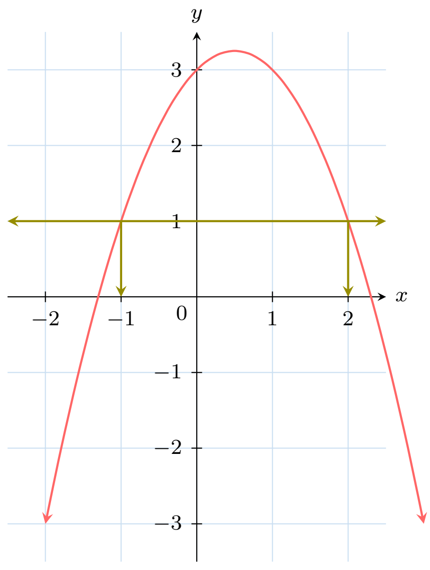
$x =$ $\boxed{-1}$ or $x =$ $\boxed{2}$

Answer:

- Draw a horizontal line at $y = 1$.

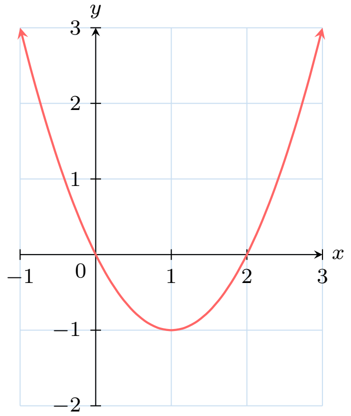


- Identify the intersection point with the curve.



$x = -1$ or $x = 2$

Ex 15: The graph of $y = f(x)$ is:

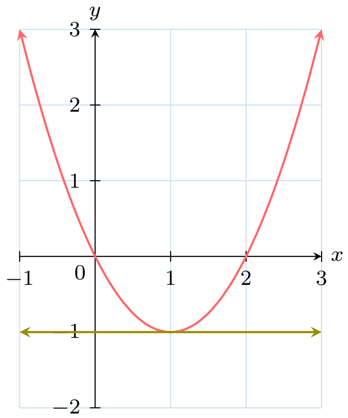


Find x such that $f(x) = -1$.

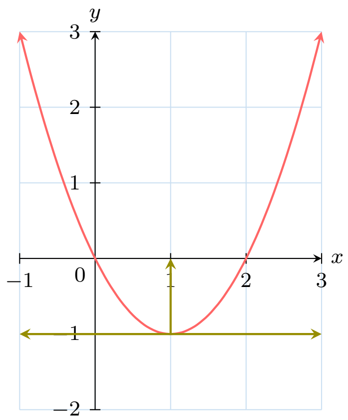
$$x = \boxed{1}$$

Answer:

- Draw a horizontal line at $y = -1$.

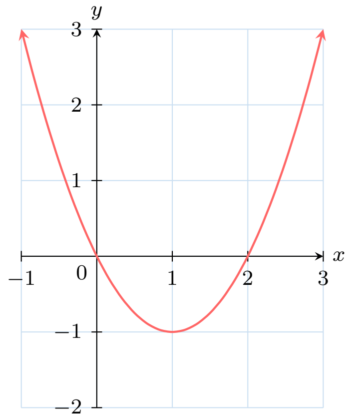


- Identify the intersection point with the curve.



$x = 1$

Ex 16: The graph of $y = f(x)$ is:

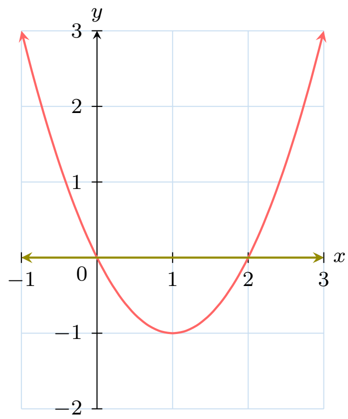


Find x such that $f(x) = 0$. These values of x are the x-**intercepts** of the graph.

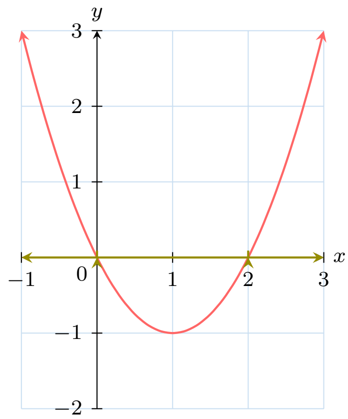
$$x = \boxed{0} \text{ or } x = \boxed{2}$$

Answer:

- Draw a horizontal line at $y = 0$.



- Identify the intersection points with the curve.



$x = 0$ or $x = 2$

C.2 FINDING x SUCH THAT $f(x) = y$ ANALYTICALLY

Ex 17: For the function $f(x) = x^2 + 2x - 2$, find the value(s) of x for which $f(x) = 1$.

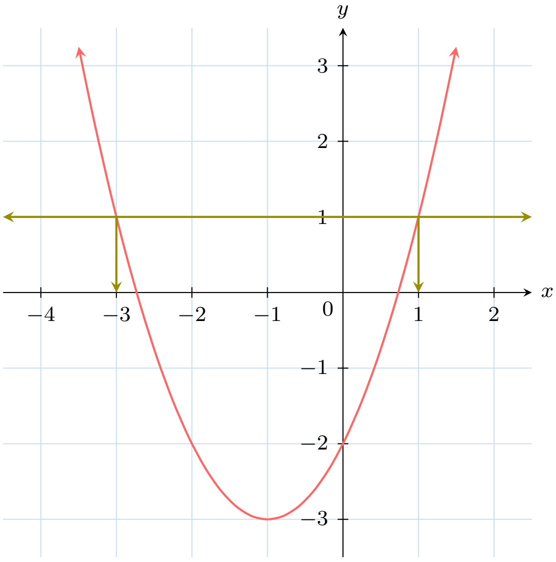
Answer: For $f(x) = 1$, we have:

$$x^2 + 2x - 2 = 1$$
$$x^2 + 2x - 3 = 0$$

We solve the quadratic equation $x^2 + 2x - 3 = 0$, with $a = 1$, $b = 2$, $c = -3$.

1. Discriminant: $\Delta = b^2 - 4ac$
$$= (2)^2 - 4(1)(-3)$$
$$= 4 + 12$$
$$= 16$$
2. As $\Delta > 0$, there are 2 distinct real roots.
3. Solutions: $x = \frac{-b - \sqrt{\Delta}}{2a}$ or $x = \frac{-b + \sqrt{\Delta}}{2a}$
$$x = \frac{-2 - \sqrt{16}}{2 \cdot 1} \quad \text{or } x = \frac{-2 + \sqrt{16}}{2 \cdot 1}$$
$$x = \frac{-2 - 4}{2} \quad \text{or } x = \frac{-2 + 4}{2}$$
$$x = -3 \quad \text{or } x = 1$$

So the values of x for which $f(x) = 1$ are $x = -3$ and $x = 1$.



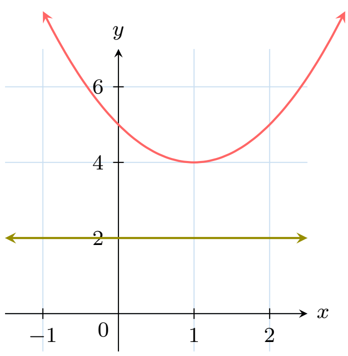
Ex 18: For the function $f(x) = x^2 - 2x + 5$, find the value(s) of x for which $f(x) = 2$.

Answer: For $f(x) = 2$, we have:
$$x^2 - 2x + 5 = 2$$
$$x^2 - 2x + 3 = 0$$

We solve the quadratic equation $x^2 - 2x + 3 = 0$, with $a = 1$, $b = -2$, $c = 3$.

1. Discriminant: $\Delta = b^2 - 4ac$
$$= (-2)^2 - 4(1)(3)$$
$$= 4 - 12$$
$$= -8$$
2. As $\Delta < 0$, there are no real roots.
3. No real solutions.

So there are no real values of x for which $f(x) = 2$.



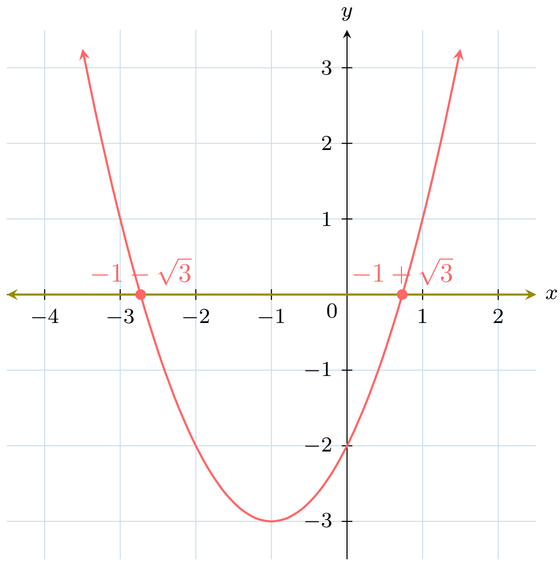
Ex 19: For the function $f(x) = x^2 + 2x - 2$, find the x-intercept(s) (the value(s) of x for which $f(x) = 0$).

Answer: For $f(x) = 0$, we have:
$$x^2 + 2x - 2 = 0$$

We solve the quadratic equation $x^2 + 2x - 2 = 0$, with $a = 1$, $b = 2$, $c = -2$.

1. Discriminant: $\Delta = b^2 - 4ac$
$$= (2)^2 - 4(1)(-2)$$
$$= 4 + 8$$
$$= 12$$
2. As $\Delta > 0$, there are 2 distinct real roots.
3. Solutions: $x = \frac{-b - \sqrt{\Delta}}{2a}$ or $x = \frac{-b + \sqrt{\Delta}}{2a}$
$$x = \frac{-2 - \sqrt{12}}{2 \cdot 1} \quad \text{or } x = \frac{-2 + \sqrt{12}}{2 \cdot 1}$$
$$x = \frac{-2 - 2\sqrt{3}}{2} \quad \text{or } x = \frac{-2 + 2\sqrt{3}}{2}$$
$$x = -1 - \sqrt{3} \quad \text{or } x = -1 + \sqrt{3}$$

So the x-intercepts are at $x = -1 - \sqrt{3}$ and $x = -1 + \sqrt{3}$.



C.3 APPLYING QUADRATIC FUNCTIONS TO REAL-WORLD SITUATIONS

Ex 20: A ball is thrown upward. Its height above the ground is given by the function
$$h(t) = -5t^2 + 20t + 1 \text{ metres,}$$
where t is the time in seconds from when the ball is thrown.

1. How high is the ball above the ground after 2 seconds?

$\boxed{21}$ m

2. From what height above the ground was the ball released?

$\boxed{1}$ m

3. At what times is the ball 16 m above the ground? (order from lowest to highest)

$\boxed{1}$ s, $\boxed{3}$ s

Answer:

1. At $t = 2$ s,
$$\begin{aligned} h(2) &= -5(2)^2 + 20(2) + 1 \\ &= -20 + 40 + 1 \\ &= 21 \end{aligned}$$
The ball is 21 m above the ground.

2. The ball was released when $t = 0$ s.
$$\begin{aligned} h(0) &= -5(0)^2 + 20(0) + 1 \\ &= 1 \end{aligned}$$
The ball was released from 1 m above ground level.

3. When $h(t) = 16$,
$$\begin{aligned} -5t^2 + 20t + 1 &= 16 \\ -5t^2 + 20t - 15 &= 0 \end{aligned}$$
With $a = -5$, $b = 20$, $c = -15$:
   - $\begin{aligned} \Delta &= b^2 - 4ac \\ &= (20)^2 - 4(-5)(-15) \\ &= 400 - 300 \\ &= 100 \end{aligned}$
   - As $\Delta > 0$, there are 2 distinct roots.
   - $\begin{aligned} t &= \frac{-b - \sqrt{\Delta}}{2a} && \text{or } t = \frac{-b + \sqrt{\Delta}}{2a} \\ t &= \frac{-20 - \sqrt{100}}{2(-5)} && \text{or } t = \frac{-20 + \sqrt{100}}{2(-5)} \\ t &= \frac{-20 - 10}{-10} && \text{or } t = \frac{-20 + 10}{-10} \\ t &= \frac{-30}{-10} && \text{or } t = \frac{-10}{-10} \\ t &= 3 && \text{or } t = 1 \end{aligned}$

   The ball is 16 m above the ground after 1 second and after 3 seconds.

Ex 21: A manufacturer produces x cakes. The profit from producing x cakes is given by the function
$$P(x) = -5x^2 + 30x + 2 \text{ dollars},$$
where x is the number of cakes produced.

1. What is the profit from producing 3 cakes?

   $\boxed{47}$ dollars

2. What is the fixed profit (or loss) when no cakes are produced?

   $\boxed{2}$ dollars

3. For what numbers of cakes is the profit 27 dollars? (order from lowest to highest)

   $\boxed{1}$, $\boxed{5}$

Answer:

1. For $x = 3$,
$$\begin{aligned} P(3) &= -5(3)^2 + 30(3) + 2 \\ &= -45 + 90 + 2 \\ &= 47 \end{aligned}$$
The profit is 47 dollars.

2. When no cakes are produced, $x = 0$.
$$\begin{aligned} P(0) &= -5(0)^2 + 30(0) + 2 \\ &= 2 \end{aligned}$$
The fixed profit is 2 dollars.

3. When $P(x) = 27$,
$$\begin{aligned} -5x^2 + 30x + 2 &= 27 \\ -5x^2 + 30x - 25 &= 0 \end{aligned}$$
With $a = -5$, $b = 30$, $c = -25$:
   - $\begin{aligned} \Delta &= b^2 - 4ac \\ &= (30)^2 - 4(-5)(-25) \\ &= 900 - 500 \\ &= 400 \end{aligned}$
   - As $\Delta > 0$, there are 2 distinct roots.
   - $\begin{aligned} x &= \frac{-b - \sqrt{\Delta}}{2a} && \text{or } x = \frac{-b + \sqrt{\Delta}}{2a} \\ x &= \frac{-30 - \sqrt{400}}{2(-5)} && \text{or } x = \frac{-30 + \sqrt{400}}{2(-5)} \\ x &= \frac{-30 - 20}{-10} && \text{or } x = \frac{-30 + 20}{-10} \\ x &= \frac{-50}{-10} && \text{or } x = \frac{-10}{-10} \\ x &= 5 && \text{or } x = 1 \end{aligned}$

   The profit is 27 dollars when 1 or 5 cakes are produced.

Ex 22: A stone is thrown into the air. Its height above the ground is given by the function
$$h(t) = -5t^2 + 30t + 2 \text{ metres},$$
where t is the time in seconds from when the stone is thrown.

1. How high is the stone above the ground after 3 seconds?

   $\boxed{47}$ m

2. From what height above the ground was the stone released?

   $\boxed{2}$ m

3. At what times is the stone 27 m above the ground? (order from lowest to highest)

   $\boxed{1}$ s, $\boxed{5}$ s

Answer:

1. At $t = 3$ s,
$$\begin{aligned} h(3) &= -5(3)^2 + 30(3) + 2 \\ &= -45 + 90 + 2 \\ &= 47 \end{aligned}$$
The stone is 47 m above the ground.

2. The stone was released when $t = 0$ s.
$$\begin{aligned} h(0) &= -5(0)^2 + 30(0) + 2 \\ &= 2 \end{aligned}$$
The stone was released from 2 m above ground level.

3. When $h(t) = 27$,
$$\begin{aligned} -5t^2 + 30t + 2 &= 27 \\ -5t^2 + 30t - 25 &= 0 \end{aligned}$$
With $a = -5$, $b = 30$, $c = -25$:
   - $\Delta = b^2 - 4ac$
     $= (30)^2 - 4(-5)(-25)$
     $= 900 - 500$
     $= 400$
   - As $\Delta > 0$, there are 2 distinct roots.
   - $t = \dfrac{-b - \sqrt{\Delta}}{2a}$ or $t = \dfrac{-b + \sqrt{\Delta}}{2a}$
     $t = \dfrac{-30 - \sqrt{400}}{2(-5)}$ or $t = \dfrac{-30 + \sqrt{400}}{2(-5)}$
     $t = \dfrac{-30 - 20}{-10}$ or $t = \dfrac{-30 + 20}{-10}$
     $t = \dfrac{-50}{-10}$ or $t = \dfrac{-10}{-10}$
     $t = 5$ or $t = 1$

   The stone is 27 m above the ground after 1 second and after 5 seconds.