REAL POLYNOMIALS

Linear and quadratic functions are the first two members of a broader class of functions called **polynomials**. Polynomials are fundamental in mathematics and are used to model a vast range of phenomena, from the trajectory of a projectile to complex economic trends. In this chapter, we will explore the algebra of polynomials, including their operations, factors, roots, and key theorems that govern their behavior.

A DEFINITIONS

Definition Polynomial

- A **polynomial in the variable x** is an algebraic expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

  where n is a non-negative integer and $a_0, a_1, \ldots, a_n$ are constants called the **coefficients**.

- A **polynomial equation in x** is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

  with $a_n \neq 0$ and $n \geq 1$.

- A **polynomial function** is a function defined for all real numbers x by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

We require that $a_n \neq 0$:

- The **degree** of the polynomial is the highest power of the variable x that has a non-zero coefficient; this exponent is n.
- The **leading coefficient** is the coefficient of the term with the highest power, a_n.
- The **constant term** is the coefficient without a variable, a_0.

A **real polynomial** is a polynomial for which all coefficients satisfy $a_i \in \mathbb{R}$.

Ex: For the polynomial function $P(x) = 2x^4 + 3x^2 + 5x + 4$, find the degree, the leading coefficient, the coefficient of x, and the constant term.

Answer: We can imagine the missing x^3 term as $0x^3$ and write

$$P(x) = 2x^4 + 0x^3 + 3x^2 + 5x + 4.$$

- The **degree** is the highest power of the variable x with non-zero coefficient. Here, the highest power is **4**.
- The **leading coefficient** is the coefficient of the term with the highest power, which is $2x^4$. Therefore, the leading coefficient is **2**.
- The **coefficient of** x is the constant multiplying the term x (or x^1). In this case, the term is $5x$, so the coefficient is **5**.
- The **constant term** is the term without a variable. Here, it is **4**.

Definition Names of Low-Degree Polynomials

Polynomials of low degree have special names:

Polynomial function	Degree	Name
$ax + b, \quad a \neq 0$	1	**linear**
$ax^2 + bx + c, \quad a \neq 0$	2	**quadratic**
$ax^3 + bx^2 + cx + d, \quad a \neq 0$	3	**cubic**
$ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$	4	**quartic**

Definition Equality of Polynomials

Two polynomials are **equal** if the coefficients of each corresponding power of the variable are identical (that is, the coefficient of x^k is the same in both polynomials for every k).

This principle is fundamental as it allows us to find unknown constants by equating the coefficients of terms with the same power. This technique is known as **identification of coefficients**.

Ex: Find the coefficients a and b given $ax + b = 2x - 3$.

Answer: By equating the coefficients of corresponding powers of x:

- The coefficients of the x term must be equal: $a = 2$.
- The constant terms must be equal: $b = -3$.

B OPERATIONS WITH POLYNOMIALS

Method Operations with Polynomials

Polynomials can be added, subtracted, and multiplied. The key principle for addition and subtraction is to combine **like terms**: terms that have the same power of the variable x.

Ex: For $P(x) = 4x^3 + 2x^2 - 5x + 1$ and $Q(x) = x^3 - 3x^2 + 7$, find:

1. $P(x) + Q(x)$
2. $P(x) - Q(x)$

Answer:

1. **Addition**: We group like terms.

$$\begin{aligned} P(x) + Q(x) &= (4x^3 + 2x^2 - 5x + 1) + (x^3 - 3x^2 + 7) \\ &= (4x^3 + x^3) + (2x^2 - 3x^2) - 5x + (1 + 7) \\ &= 5x^3 - x^2 - 5x + 8 \end{aligned}$$

2. **Subtraction**: Be careful with the signs when distributing the negative.

$$\begin{aligned} P(x) - Q(x) &= (4x^3 + 2x^2 - 5x + 1) - (x^3 - 3x^2 + 7) \\ &= 4x^3 + 2x^2 - 5x + 1 - x^3 + 3x^2 - 7 \\ &= (4x^3 - x^3) + (2x^2 + 3x^2) - 5x + (1 - 7) \\ &= 3x^3 + 5x^2 - 5x - 6 \end{aligned}$$

Ex: For $P(x) = x^3 - 2x + 4$ and $Q(x) = 2x^2 + 3x - 5$, find $P(x)Q(x)$.

Answer: To multiply two polynomials, we multiply each term of the first polynomial by each term of the second polynomial.

$$\begin{aligned} P(x)Q(x) &= (x^3 - 2x + 4)(2x^2 + 3x - 5) \\ &= x^3(2x^2 + 3x - 5) - 2x(2x^2 + 3x - 5) + 4(2x^2 + 3x - 5) \\ &= (2x^5 + 3x^4 - 5x^3) - (4x^3 + 6x^2 - 10x) + (8x^2 + 12x - 20) \\ &= 2x^5 + 3x^4 - 5x^3 - 4x^3 - 6x^2 + 10x + 8x^2 + 12x - 20 \\ &= 2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20 \end{aligned}$$

The degree of the product of two non-zero polynomials $P(x)$ and $Q(x)$ is the sum of their individual degrees. This is because the leading term of the product $P(x)Q(x)$ is obtained by multiplying the leading term of $P(x)$ by the leading term of $Q(x)$, and when we multiply these terms, their exponents add.
In this specific case, since the degree of $P(x)$ is 3 and the degree of $Q(x)$ is 2, the degree of their product is $3 + 2 = 5$.

C THE DIVISION ALGORITHM

Proposition Division With Remainder

If a polynomial P is divided by a non-zero polynomial D, then there exist unique polynomials Q and R such that

$$P = DQ + R \quad \text{with } \deg(R) < \deg(D).$$

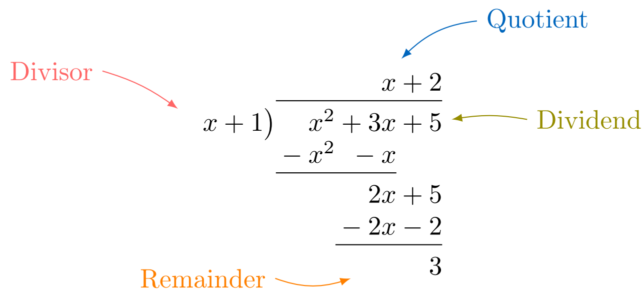
Here P is the **dividend**, D is the **divisor**, Q is the **quotient**, and R is the **remainder**. The remainder R may be the zero polynomial.

Method Division Algorithm

The division of one polynomial by another can be performed using a **long division algorithm**, which is analogous to the method used for dividing integers.

Ex:

$$\begin{array}{ccccccccc} x^2+3x+5 & = & (x+1) & \times & (x+2) & + & 3 & \text{with} & \deg(3)<\deg(x+1) \\ \text{Dividend} & = & \text{Divisor} & \times & \text{Quotient} & + & \text{Remainder} & \text{with} & \deg(\text{Remainder})<\deg(\text{Divisor}) \end{array}$$



The Division Algorithm describes the result of any polynomial division. A special and important case is when the division is **exact**, meaning there is no remainder ($R = 0$). This leads to the concept of a divisor, or factor.

Definition Divisor of a Polynomial

Let P and D be polynomials. We say that D **divides** P if there exists a polynomial Q such that

$$P(x) = D(x)Q(x).$$

In this case, D is called a **divisor** or a **factor** of P.

Proposition Condition for Divisibility

A polynomial D is a divisor of a polynomial P if and only if the remainder of the division of P by D is the zero polynomial ($R = 0$).

Proof

We must prove both directions of the statement.

- **($\Rightarrow$): If D is a divisor of P, then the remainder is zero.**
  Assume that D is a divisor of P.
  By the definition of a divisor, this means there exists a polynomial Q such that

  $$P(x) = D(x)Q(x).$$

  This can be rewritten as $P(x) = D(x)Q(x) + 0$.
  By the Division Algorithm, there is a unique quotient and remainder. Comparing this with the standard form $P(x) = D(x)Q(x) + R(x)$, we can see by uniqueness that the remainder $R(x)$ must be 0.

- **($\Leftarrow$): If the remainder is zero, then D is a divisor of P.**
  Assume that the remainder of the division of P by D is zero.
  By the Division Algorithm, we can write

  $$P(x) = D(x)Q(x) + R(x).$$

  Substituting $R(x) = 0$ into this equation, we get

  $$\begin{aligned} P(x) &= D(x)Q(x) + 0 \\ &= D(x)Q(x). \end{aligned}$$

  This is precisely the definition of D being a divisor of P.

D THE REMAINDER AND FACTOR THEOREMS

While the Division Algorithm applies to any polynomial divisor, two powerful theorems emerge when we consider the specific case of dividing by a linear polynomial of the form $(x - k)$.

Proposition The Remainder Theorem

When a polynomial P is divided by a linear polynomial of the form $(x-k)$, the remainder is the constant value $P(k)$.

Proof

From the Division Algorithm, when a polynomial $P(x)$ is divided by the divisor $D(x) = x - k$, there exists a unique quotient $Q(x)$ and a remainder $R(x)$ such that

$$P(x) = (x-k)Q(x) + R(x).$$

We know that the degree of the remainder must be less than the degree of the divisor. Since the degree of the divisor $(x-k)$ is 1, the degree of the remainder R must be 0. This means that the remainder is a constant; let's call this constant r.
The division equation can therefore be written as

$$P(x) = (x-k)Q(x) + r.$$

This identity is true for all values of x. If we substitute $x = k$ into the equation, we get

$$\begin{aligned} P(k) &= (k-k)Q(k) + r \\ &= 0 \cdot Q(k) + r \\ &= r. \end{aligned}$$

Thus, the remainder r is precisely the value of the polynomial when evaluated at k: $r = P(k)$.

Definition Root

Let P be a polynomial.
A number α is a **root** (or **zero**) of P if $P(\alpha) = 0$.

Proposition The Factor Theorem

For any polynomial P, $(x-\alpha)$ is a factor of P if and only if $P(\alpha) = 0$.

Proof

Let $P(x)$ be a polynomial and α be a number.

$$\begin{aligned} & (x-\alpha) \text{ is a factor of } P(x) & \\ \iff & \text{the remainder of the division of } P(x) \text{ by } (x-\alpha) \text{ is } 0 & \text{(by the Condition for Divisibility)} \\ \iff & P(\alpha) = 0 & \text{(by the Remainder Theorem, since the remainder is } P(\alpha)\text{)} \end{aligned}$$

Proposition Number of Roots

A polynomial P of degree $n \geq 0$ has at most n distinct roots.

Proof

We proceed by induction on the degree n of the polynomial.

- **Base case ($n = 0$):** A polynomial of degree 0 is a non-zero constant ($P(x) = a_0$ with $a_0 \neq 0$). It has 0 roots. Since $0 \leq 0$, the property holds.
- **Inductive step:** Suppose that every polynomial of degree n has at most n roots. Let P be a polynomial of degree $n+1$.
  - If P has no roots, then $0 \leq n+1$, and the property holds.
  - If P has at least one root α, then by the Factor Theorem, we can write:

    $$P(x) = (x-\alpha)Q(x)$$

    where Q is a polynomial of degree n. By the inductive hypothesis, Q has at most n roots. The roots of P are α plus the roots of Q. Therefore, P has at most $n+1$ roots.
- **Conclusion:** By the principle of induction, a polynomial of degree n has at most n roots.

E QUADRATIC EQUATIONS WITH COMPLEX ROOTS

The Factor Theorem connects roots to factors, but it does not guarantee that a real polynomial has any real roots (for example, $P(x) = x^2 + 1$). To fully understand the roots of all polynomials, we must consider solutions in the set of complex numbers, $\mathbb{C}$.

Proposition Solving equations of the form $x^2 = k$

Consider the equation $x^2 = k$ where k is a real number.

- If $k \geq 0$, the equation has two real roots: $x = \sqrt{k}$ and $x = -\sqrt{k}$ (these are equal if $k = 0$).
- If $k < 0$, the equation has two purely imaginary roots: $x = i\sqrt{|k|}$ and $x = -i\sqrt{|k|}$.

Consider the equation $z^2 = k$, where k is a real number.

- If $k > 0$, the equation has two distinct real solutions:

$$z_1 = \sqrt{k} \quad \text{and} \quad z_2 = -\sqrt{k}$$

- If $k = 0$, the equation has a unique real solution:

$$z_0 = 0$$

- If $k < 0$, the equation has two conjugate purely imaginary solutions:

$$z_1 = i\sqrt{-k} \quad \text{and} \quad z_2 = -i\sqrt{-k}$$

Proposition Solving Quadratic Equations with Real Coefficients in $\mathbb{C}$

Consider the equation $az^2 + bz + c = 0$, where a, b, and c are real numbers and $a \neq 0$.
Let $\Delta = b^2 - 4ac$ be the discriminant of this equation.

- If $\Delta > 0$, the equation has two distinct real solutions:

$$z_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{and} \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

- If $\Delta = 0$, the equation has a unique real solution:

$$z_0 = \frac{-b}{2a}$$

- If $\Delta < 0$, the equation has two conjugate complex solutions:

$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a} \quad \text{and} \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a}$$

- If $\Delta \neq 0$, the quadratic can be factorized as: $az^2 + bz + c = a(z - z_1)(z - z_2)$.