SEQUENCES

A NUMERICAL SEQUENCE

A.1 FINDING u_n

Ex 1:

n	1	2	3	4	5	6
u_n	3	5	7	9	11	13

What is u_4?

$\boxed{9}$

Answer: $u_4 = 9$.

Ex 2:

n	1	2	3	4	5	6
u_n	2	6	12	20	30	42

What is u_5?

$\boxed{30}$

Answer: $u_5 = 30$.

Ex 3:

n	1	2	3	4	5	6	7	8
u_n	4	9	16	25	36	49	64	81

What is u_7?

$\boxed{64}$

Answer: $u_7 = 64$.

Ex 4:

n	1	2	3	4	5	6	7	8
u_n	1	3	7	15	31	63	127	255

What is u_8?

$\boxed{255}$

Answer: $u_8 = 255$.

A.2 FINDING u_n IN AN ARITHMETIC SEQUENCE

Ex 5: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	3	5	7	9	11	$\boxed{13}$

Answer: $u_6 = 13$, because each term increases by 2.

n	1	2	3	4	5	6
u_n	3	5	7	9	11	13

+2 +2 +2 +2 +2

Ex 6: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	3	8	13	18	23	$\boxed{28}$

Answer: $u_6 = 28$, because each term increases by 5.

n	1	2	3	4	5	6
u_n	3	8	13	18	23	28

+5 +5 +5 +5 +5

Ex 7: What is u_5 for this sequence?

n	1	2	3	4	5
u_n	20	18	16	14	$\boxed{12}$

Answer: $u_5 = 12$, because each term decreases by 2.

n	1	2	3	4	5
u_n	20	18	16	14	12

−2 −2 −2 −2

Ex 8: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	80	70	60	50	40	$\boxed{30}$

Answer: $u_6 = 30$, because each term decreases by 10.

n	1	2	3	4	5	6
u_n	80	70	60	50	40	30

−10 −10 −10 −10 −10

B DEFINITION USING A RECURSIVE RULE

B.1 CALCULATING THE FIRST TERMS

Ex 9: Write the sequence defined by: the first term is 7, and each term is obtained by adding 4 to the previous term.

$(\boxed{7}, \boxed{11}, \boxed{15}, \boxed{19}, \boxed{23}, \dots)$

Answer:

$$7 \xrightarrow{+4} 11 \xrightarrow{+4} 15 \xrightarrow{+4} 19 \xrightarrow{+4} 23$$

The sequence is: $(7, 11, 15, 19, 23, \dots)$.

Ex 10: Write the sequence defined by: the first term is 1, and each term is obtained by multiplying the previous term by 2.

$(\boxed{1}, \boxed{2}, \boxed{4}, \boxed{8}, \boxed{16}, \dots)$

Answer:

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16$$

The sequence is: $(1, 2, 4, 8, 16, \dots)$.

Ex 11: Write the sequence defined by: the first term is 10, and each term is obtained by subtracting 5 from the previous term.

$$(\boxed{10}, \boxed{5}, \boxed{0}, \boxed{-5}, \boxed{-10}, \dots)$$

Answer:

$$10 \xrightarrow{-5} 5 \xrightarrow{-5} 0 \xrightarrow{-5} -5 \xrightarrow{-5} -10$$

The sequence is: $(10, 5, 0, -5, -10, \dots)$.

Ex 12: Write the sequence defined by: the first term is 2.5, and each term is obtained by adding 0.5 to the previous term.

$$(\boxed{2.5}, \boxed{3}, \boxed{3.5}, \boxed{4}, \boxed{4.5}, \dots)$$

Answer:

$$2.5 \xrightarrow{+0.5} 3 \xrightarrow{+0.5} 3.5 \xrightarrow{+0.5} 4 \xrightarrow{+0.5} 4.5$$

The sequence is: $(2.5, 3, 3.5, 4, 4.5, \dots)$.

B.2 CALCULATING THE FIRST TERMS

Ex 13: Calculate the first terms of the sequence defined by:

$$u_0 = 3 \text{ and } u_{n+1} = u_n + 4.$$

- $u_1 = \boxed{7}$
- $u_2 = \boxed{11}$
- $u_3 = \boxed{15}$
- $u_4 = \boxed{19}$
- $u_5 = \boxed{23}$

Answer:

$$3 \xrightarrow{+4} 7 \xrightarrow{+4} 11 \xrightarrow{+4} 15 \xrightarrow{+4} 19 \xrightarrow{+4} 23$$

- $u_1 = u_0 + 4 = 3 + 4 = 7$
- $u_2 = u_1 + 4 = 7 + 4 = 11$
- $u_3 = u_2 + 4 = 11 + 4 = 15$
- $u_4 = u_3 + 4 = 15 + 4 = 19$
- $u_5 = u_4 + 4 = 19 + 4 = 23$

Ex 14: Calculate the first terms of the sequence defined by:

$$u_0 = 3 \text{ and } u_{n+1} = 2u_n.$$

- $u_1 = \boxed{6}$
- $u_2 = \boxed{12}$
- $u_3 = \boxed{24}$
- $u_4 = \boxed{48}$
- $u_5 = \boxed{96}$

Answer:

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{\times 2} 48 \xrightarrow{\times 2} 96$$

- $u_1 = 2u_0 = 2 \times 3 = 6$
- $u_2 = 2u_1 = 2 \times 6 = 12$
- $u_3 = 2u_2 = 2 \times 12 = 24$
- $u_4 = 2u_3 = 2 \times 24 = 48$
- $u_5 = 2u_4 = 2 \times 48 = 96$

Ex 15: Calculate the first terms of the sequence defined by:

$$u_0 = 12 \text{ and } u_{n+1} = u_n - 10.$$

- $u_1 = \boxed{2}$
- $u_2 = \boxed{-8}$
- $u_3 = \boxed{-18}$
- $u_4 = \boxed{-28}$
- $u_5 = \boxed{-38}$

Answer:

$$12 \xrightarrow{-10} 2 \xrightarrow{-10} -8 \xrightarrow{-10} -18 \xrightarrow{-10} -28 \xrightarrow{-10} -38$$

- $u_1 = u_0 - 10 = 12 - 10 = 2$
- $u_2 = u_1 - 10 = 2 - 10 = -8$
- $u_3 = u_2 - 10 = -8 - 10 = -18$
- $u_4 = u_3 - 10 = -18 - 10 = -28$
- $u_5 = u_4 - 10 = -28 - 10 = -38$

Ex 16: Calculate the first terms of the sequence defined by:

$$u_0 = 64 \text{ and } u_{n+1} = \frac{u_n}{2}.$$

- $u_1 = \boxed{32}$
- $u_2 = \boxed{16}$
- $u_3 = \boxed{8}$
- $u_4 = \boxed{4}$
- $u_5 = \boxed{2}$

Answer:

$$64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2$$

- $u_1 = \frac{u_0}{2} = \frac{64}{2} = 32$
- $u_2 = \frac{u_1}{2} = \frac{32}{2} = 16$
- $u_3 = \frac{u_2}{2} = \frac{16}{2} = 8$
- $u_4 = \frac{u_3}{2} = \frac{8}{2} = 4$
- $u_5 = \frac{u_4}{2} = \frac{4}{2} = 2$

B.3 IDENTIFYING THE RECURSIVE RULE

Ex 17: Given the sequence: $(3, 5, 7, 9, 11, 13, \dots)$

- The first term is $\boxed{3}$.
- The rule is $\boxed{\textbf{Add}}$ $\boxed{2}$

Answer:

- The first term is 3.
- The rule is add 2:

$$3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \xrightarrow{+2} 11 \xrightarrow{+2} 13$$

Ex 18: Given the sequence: $(60, 55, 50, 45, 40, 35, \dots)$

- The first term is $\boxed{60}$.
- The rule is $\boxed{\textbf{Subtract}}$ $\boxed{5}$

Answer:

- The first term is 60.
- The rule is subtract 5:

$$60 \xrightarrow{-5} 55 \xrightarrow{-5} 50 \xrightarrow{-5} 45 \xrightarrow{-5} 40 \xrightarrow{-5} 35$$

Ex 19: Given the sequence: $(64, 32, 16, 8, 4, 2, \dots)$

- The first term is $\boxed{64}$.
- The rule is $\boxed{\textbf{Divide}}$ $\boxed{2}$

Answer:

- The first term is 64.
- The rule is divide by 2:

$$64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2$$

Ex 20: Given the sequence: $(1, 10, 100, 1\,000, 10\,000, \dots)$

- The first term is $\boxed{1}$.
- The rule is $\boxed{\textbf{Multiply}}$ $\boxed{10}$

Answer:

- The first term is 1.
- The rule is multiply by 10:

$$1 \xrightarrow{\times 10} 10 \xrightarrow{\times 10} 100 \xrightarrow{\times 10} 1\,000 \xrightarrow{\times 10} 10\,000$$

B.4 IDENTIFYING THE RECURSIVE RULE

Ex 21: Given the sequence: $(3, 5, 7, 9, 11, 13, \dots)$, what is its recursive rule?

- $u_0 = \boxed{3}$.
- $u_{n+1} = \boxed{u_n + 2}$

Answer:

- The first term is $u_0 = 3$.
- The rule is add 2:

$$(\ 3 \overset{+2}{\frown}\ 5 \overset{+2}{\frown}\ 7 \overset{+2}{\frown}\ 9\ , \dots , u_n \overset{+2}{\frown}\ u_{n+1}\ , \dots)$$

$$u_{n+1} = u_n + 2$$

Ex 22: Given the sequence: $(100, 90, 80, 70, 60, \dots)$, what is its recursive rule?

- $u_0 = \boxed{100}$.
- $u_{n+1} = \boxed{u_n - 10}$

Answer:

- The first term is $u_0 = 100$.
- The rule is subtract 10:

$$(\ 100 \overset{-10}{\frown}\ 90 \overset{-10}{\frown}\ 80 \overset{-10}{\frown}\ 70\ , \dots , u_n \overset{-10}{\frown}\ u_{n+1}\ , \dots)$$

$$u_{n+1} = u_n - 10$$

Ex 23: Given the sequence: $(2, 6, 18, 54, 162, \dots)$, what is its recursive rule?

- $u_0 = \boxed{2}$.
- $u_{n+1} = \boxed{3 \times u_n}$

Answer:

- The first term is $u_0 = 2$.
- The rule is multiply by 3:

$$(\ 2 \overset{\times 3}{\frown}\ 6 \overset{\times 3}{\frown}\ 18 \overset{\times 3}{\frown}\ 54\ , \dots , u_n \overset{\times 3}{\frown}\ u_{n+1}\ , \dots)$$

$$u_{n+1} = 3 \times u_n$$

Ex 24: Given the sequence: $(8, 4, 2, 1, 0.5, 0.25, \dots)$, what is its recursive rule?

- $u_0 = \boxed{8}$.
- $u_{n+1} = \boxed{u_n \div 2}$

Answer:

- The first term is $u_0 = 8$.
- The rule is divide by 2:

$$(\ 8 \overset{\div 2}{\frown}\ 4 \overset{\div 2}{\frown}\ 2 \overset{\div 2}{\frown}\ 1 \overset{\div 2}{\frown}\ 0.5\ , \dots , u_n \overset{\div 2}{\frown}\ u_{n+1}\ , \dots)$$

$$u_{n+1} = \frac{u_n}{2}$$

B.5 MODELING REAL SITUATIONS WITH SEQUENCES

Ex 25: A scientist observes a culture of bacteria. At the start, there are $u_0 = 5$ bacteria in a petri dish. Each day, the number of bacteria doubles.
Let u_n be the number of bacteria at the day n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{10}$ bacteria
- $u_2 = \boxed{20}$ bacteria
- $u_3 = \boxed{40}$ bacteria

What is its recursive rule?

$$u_{n+1} = \boxed{2 \times u_n}$$

Answer: The number of bacteria doubles each day:

- $u_1 = 2 \times u_0 = 2 \times 5 = 10$
- $u_2 = 2 \times u_1 = 2 \times 10 = 20$
- $u_3 = 2 \times u_2 = 2 \times 20 = 40$

The rule is multiply by 2:

$$u_{n+1} = 2 \times u_n$$

Ex 26: Each day, I walk more and more steps. On day 0, I walk $u_0 = 1000$ steps. Each day, I walk 500 steps more than the day before.
Let u_n be the number of steps I have walked at the end of day n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{1500}$ steps
- $u_2 = \boxed{2000}$ steps
- $u_3 = \boxed{2500}$ steps

What is its recursive rule?

$$u_{n+1} = \boxed{u_n + 500}$$

Answer: The number of steps increases by 500 each day:

- $u_1 = u_0 + 500 = 1000 + 500 = 1500$
- $u_2 = u_1 + 500 = 1500 + 500 = 2000$
- $u_3 = u_2 + 500 = 2000 + 500 = 2500$

The recursive rule is:

$$u_{n+1} = u_n + 500$$

Ex 27: Suppose I deposit \$100 in a savings account. Each year, my amount is multiplied by 1.1 (that is, it increases by 10% each year).
Let u_n be the amount of money in the account after n years. What are the first three terms of the sequence (u_n)?

- $u_0 = \boxed{100}$ dollars
- $u_1 = \boxed{110}$ dollars
- $u_2 = \boxed{121}$ dollars

What is its recursive rule?

$$u_{n+1} = \boxed{1.1 \times u_n}$$

Answer: The amount increases by 10% each year, so it is multiplied by 1.1:

- $u_0 = 100$
- $u_1 = 1.1 \times u_0 = 1.1 \times 100 = 110$
- $u_2 = 1.1 \times u_1 = 1.1 \times 110 = 121$

The recursive rule is:

$$u_{n+1} = 1.1 \times u_n$$

Ex 28: At the start, I have $u_0 = 20$ dollars. Each week, my parents give me \$10 more.
Let u_n be the amount of money I have at the beginning of week n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{30}$ dollars
- $u_2 = \boxed{40}$ dollars
- $u_3 = \boxed{50}$ dollars

What is its recursive rule?

$$u_{n+1} = \boxed{u_n + 10}$$

Answer: The sequence increases by \$10 each week:

- $u_1 = u_0 + 10 = 20 + 10 = 30$
- $u_2 = u_1 + 10 = 30 + 10 = 40$
- $u_3 = u_2 + 10 = 40 + 10 = 50$

The recursive rule is:

$$u_{n+1} = u_n + 10$$

B.6 MODELING REAL SITUATIONS WITH SEQUENCES

Ex 29: A company has 200 employees in 2025. Each year, 10% of the employees leave the company, and the company hires 30 new employees.
Let (u_n) be the sequence corresponding to the number of employees in the company in $2025 + n$.

1. How many employees will there be in 2026?

$$\boxed{210}$$

2. How many employees will there be in 2027?

$$\boxed{219}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{0.9u_n + 30}$$

Answer:

1. In 2026, which corresponds to $n = 1$:

$$\begin{aligned} u_1 &= \overbrace{u_0}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_0}^{\text{10\% who leave}} + \overbrace{30}^{\text{hired}} \\ &= 200 - 0.1 \times 200 + 30 \\ &= 200 - 20 + 30 \\ &= 210 \end{aligned}$$

There will therefore be 210 employees in 2026.

2. In 2027, which corresponds to $n = 2$:

$$\begin{aligned} u_2 &= \overbrace{u_1}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_1}^{\text{10\% who leave}} + \overbrace{30}^{\text{hired}} \\ &= 210 - 0.1 \times 210 + 30 \\ &= 210 - 21 + 30 \\ &= 219 \end{aligned}$$

There will therefore be 219 employees in 2027.

3. The recursive relation is

$$\begin{aligned} u_{n+1} &= \overbrace{u_n}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_n}^{\text{10\% who leave}} + \overbrace{30}^{\text{hired}} \\ &= u_n - 0.1 \times u_n + 30 \\ &= (1 - 0.1) \times u_n + 30 \\ &= 0.9u_n + 30 \end{aligned}$$

$u_{n+1} = 0.9u_n + 30.$

Ex 30: A gym has 200 members in 2025. Each year, the number of members increases by 10% through referrals, and the gym adds 20 new members from advertising.
Let (u_n) be the sequence corresponding to the number of members in the gym in $2025 + n$.

1. How many members will there be in 2026?

$$\boxed{240}$$

2. How many members will there be in 2027?

$$\boxed{284}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{1.1u_n + 20}$$

Answer:

1. In 2026, which corresponds to $n = 1$:

$$\begin{aligned} u_1 &= \overbrace{u_0}^{\text{members from the previous year}} + \overbrace{0.1 \times u_0}^{\text{10\% through referrals}} + \overbrace{20}^{\text{from advertising}} \\ &= 200 + 0.1 \times 200 + 20 \\ &= 200 + 20 + 20 \\ &= 240 \end{aligned}$$

There will therefore be 240 members in 2026.

2. In 2027, which corresponds to $n = 2$:

$$\begin{aligned} u_2 &= \overbrace{u_1}^{\text{members from the previous year}} + \overbrace{0.1 \times u_1}^{\text{10\% through referrals}} + \overbrace{20}^{\text{from advert}} \\ &= 240 + 0.1 \times 240 + 20 \\ &= 240 + 24 + 20 \\ &= 284 \end{aligned}$$

There will therefore be 284 members in 2027.

3. The recursive relation is

$$\begin{aligned} u_{n+1} &= \overbrace{u_n}^{\text{members from the previous year}} + \overbrace{0.1 \times u_n}^{\text{10\% through referrals}} + \overbrace{2}^{\text{from ad}} \\ &= u_n + 0.1 \times u_n + 20 \\ &= (1 + 0.1) \times u_n + 20 \\ &= 1.1u_n + 20 \end{aligned}$$

$u_{n+1} = 1.1u_n + 20.$

Ex 31: A school has 150 students in 2025. Each year, 20% of the students leave the school, and the school admits 50 new students.
Let (u_n) be the sequence corresponding to the number of students in the school in $2025 + n$.

1. How many students will there be in 2026?

$$\boxed{170}$$

2. How many students will there be in 2027?

$$\boxed{186}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{0.8u_n + 50}$$

Answer:

1. In 2026, which corresponds to $n = 1$:

$$\begin{aligned} u_1 &= \overbrace{u_0}^{\text{students from the previous year}} - \overbrace{0.2 \times u_0}^{\text{20\% who leave}} + \overbrace{50}^{\text{admitted}} \\ &= 150 - 0.2 \times 150 + 50 \\ &= 150 - 30 + 50 \\ &= 170 \end{aligned}$$

There will therefore be 170 students in 2026.

2. In 2027, which corresponds to $n = 2$:

$$\begin{aligned} u_2 &= \overbrace{u_1}^{\text{students from the previous year}} - \overbrace{0.2 \times u_1}^{\text{20\% who leave}} + \overbrace{50}^{\text{admitted}} \\ &= 170 - 0.2 \times 170 + 50 \\ &= 170 - 34 + 50 \\ &= 186 \end{aligned}$$

There will therefore be 186 students in 2027.

3. The recursive relation is

$$\begin{aligned}u_{n+1} &= \overbrace{u_n}^{\text{students from the previous year}} - \overbrace{0.2 \times u_n}^{\text{20\% who leave}} + \overbrace{50}^{\text{admitted}} \\ &= u_n - 0.2 \times u_n + 50 \\ &= 0.8u_n + 50\end{aligned}$$

$u_{n+1} = 0.8u_n + 50.$

Ex 32: A YouTube channel has 500 subscribers in 2025. Each year, the number of subscribers increases by 10% due to organic growth, and it gains 100 new subscribers from promotions.
Let (u_n) be the sequence corresponding to the number of subscribers to the channel in $2025 + n$.

1. How many subscribers will there be in 2026?

$$\boxed{650}$$

2. How many subscribers will there be in 2027?

$$\boxed{815}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{1.1u_n + 100}$$

Answer:

1. In 2026, which corresponds to $n = 1$:

$$\begin{aligned}u_1 &= \overbrace{u_0}^{\text{subscribers from the previous year}} + \overbrace{0.1 \times u_0}^{\text{10\% organic growth}} + \overbrace{100}^{\text{from promotions}} \\ &= 500 + 0.1 \times 500 + 100 \\ &= 500 + 50 + 100 \\ &= 650\end{aligned}$$

There will therefore be 650 subscribers in 2026.

2. In 2027, which corresponds to $n = 2$:

$$\begin{aligned}u_2 &= \overbrace{u_1}^{\text{subscribers from the previous year}} + \overbrace{0.1 \times u_1}^{\text{10\% organic growth}} + \overbrace{100}^{\text{from promotions}} \\ &= 650 + 0.1 \times 650 + 100 \\ &= 650 + 65 + 100 \\ &= 815\end{aligned}$$

There will therefore be 815 subscribers in 2027.

3. The recursive relation is

$$\begin{aligned}u_{n+1} &= \overbrace{u_n}^{\text{subscribers from the previous year}} + \overbrace{0.1 \times u_n}^{\text{10\% organic growth}} + \overbrace{100}^{\text{from promotions}} \\ &= u_n + 0.1 \times u_n + 100 \\ &= (1 + 0.1) \times u_n + 100 \\ &= 1.1u_n + 100\end{aligned}$$

$u_{n+1} = 1.1u_n + 100.$

C DEFINITION USING AN EXPLICIT RULE

C.1 CALCULATING TERMS FROM AN EXPLICIT FORMULA

Ex 33: Consider the sequence defined by the explicit formula: $u_n = 3n + 2$.
Write the first four terms of this sequence.

- $u_0 = \boxed{2}$
- $u_1 = \boxed{5}$
- $u_2 = \boxed{8}$
- $u_3 = \boxed{11}$

Answer:

- For $n = 0$:
$$\begin{aligned}u_0 &= 3 \times 0 + 2 \\ &= 0 + 2 \\ &= 2\end{aligned}$$
- For $n = 1$:
$$\begin{aligned}u_1 &= 3 \times 1 + 2 \\ &= 3 + 2 \\ &= 5\end{aligned}$$
- For $n = 2$:
$$\begin{aligned}u_2 &= 3 \times 2 + 2 \\ &= 6 + 2 \\ &= 8\end{aligned}$$
- For $n = 3$:
$$\begin{aligned}u_3 &= 3 \times 3 + 2 \\ &= 9 + 2 \\ &= 11\end{aligned}$$

So the first four terms are: $2,\ 5,\ 8,\ 11$.

Ex 34: Consider the sequence defined by the explicit formula: $u_n = -10n + 100$.
Write the first four terms of this sequence.

- $u_0 = \boxed{100}$
- $u_1 = \boxed{90}$
- $u_2 = \boxed{80}$
- $u_3 = \boxed{70}$

Answer:

- For $n = 0$:
$$\begin{aligned}u_0 &= -10 \times 0 + 100 \\ &= 0 + 100 \\ &= 100\end{aligned}$$
- For $n = 1$:
$$\begin{aligned}u_1 &= -10 \times 1 + 100 \\ &= -10 + 100 \\ &= 90\end{aligned}$$

- For $n = 2$:

$$\begin{aligned} u_2 &= -10 \times 2 + 100 \\ &= -20 + 100 \\ &= 80 \end{aligned}$$

- For $n = 3$:

$$\begin{aligned} u_3 &= -10 \times 3 + 100 \\ &= -30 + 100 \\ &= 70 \end{aligned}$$

So the first four terms are: $100, 90, 80, 70$.

Ex 35: Consider the sequence defined by the explicit formula: $u_n = n^2 + 2$.
Write the first four terms of this sequence.

- $u_0 = \boxed{2}$
- $u_1 = \boxed{3}$
- $u_2 = \boxed{6}$
- $u_3 = \boxed{11}$

Answer:

- For $n = 0$:

$$\begin{aligned} u_0 &= 0^2 + 2 \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

- For $n = 1$:

$$\begin{aligned} u_1 &= 1^2 + 2 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

- For $n = 2$:

$$\begin{aligned} u_2 &= 2^2 + 2 \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

- For $n = 3$:

$$\begin{aligned} u_3 &= 3^2 + 2 \\ &= 9 + 2 \\ &= 11 \end{aligned}$$

So the first four terms are: $2, 3, 6, 11$.

C.2 MODELING REAL SITUATIONS WITH SEQUENCES

Ex 36: You start with \$30 and each week your parent gives you \$10.
The amount of money you have after n weeks is given by the formula:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr weeks} \times \text{Amount received each week} \\ &= 30 + n \times 10 \\ &= 30 + 10n \end{aligned}$$

where u_n is the amount after n weeks. How much money will you have after 20 weeks?

$\boxed{230}$ dollars

Answer:

- After 20 weeks:

$$\begin{aligned} u_{20} &= 30 + 10 \times 20 \\ &= 30 + 200 \\ &= 230 \end{aligned}$$

So, after 20 weeks you will have \$230.

Ex 37: You deposit \$1 500 in a savings account that pays simple interest at a rate of 4% per year.
The amount of money in your account after n years is given by the formula:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr years} \times \text{Percentage of the initial amoun} \\ &= 1\,500 + n \times 0.04 \times 1\,500 \\ &= 1\,500 + 60n \end{aligned}$$

where u_n is the amount after n years. What is your amount at year 20?

$\boxed{2700}$ dollars

Answer:

- At year 20:

$$\begin{aligned} u_{20} &= 1\,500 + 60 \times 20 \\ &= 1\,500 + 1\,200 \\ &= 2\,700 \end{aligned}$$

So, your amount at year 20 is \$2 700.

Ex 38: You start a stamp collection with 12 stamps. Each month, you add 4 new stamps to your collection.
The number of stamps you have after n months is given by the formula:

$$\begin{aligned} u_n &= \text{Initial number of stamps} + \text{Nbr months} \times \text{Stamps added per r} \\ &= 12 + n \times 4 \\ &= 12 + 4n \end{aligned}$$

where u_n is the number of stamps after n months. How many stamps will you have after 15 months?

$\boxed{72}$ stamps

Answer:

- After 15 months:

$$\begin{aligned} u_{15} &= 12 + 4 \times 15 \\ &= 12 + 60 \\ &= 72 \end{aligned}$$

So, after 15 months you will have 72 stamps.

Ex 39: A school plants 5 trees in its garden to start. Every year, they plant 3 new trees.
The total number of trees after n years is given by the formula:

$$\begin{aligned} u_n &= \text{Initial number of trees} + \text{Nbr years} \times \text{Trees planted per year} \\ &= 5 + n \times 3 \\ &= 5 + 3n \end{aligned}$$

where u_n is the number of trees after n years. How many trees will there be after 12 years?

$\boxed{41}$ trees

Answer:

- After 12 years:

$$\begin{aligned} u_{12} &= 5 + 3 \times 12 \\ &= 5 + 36 \\ &= 41 \end{aligned}$$

So, after 12 years there will be 41 trees.

D ARITHMETIC SEQUENCES

D.1 STUDYING AN ARITHMETIC SEQUENCE

Ex 40: Consider the sequence $(5,\ 8,\ 11,\ 14,\ 17, \ldots)$

1.
   - $u_1 - u_0 = \boxed{3}$
   - $u_2 - u_1 = \boxed{3}$
   - $u_3 - u_2 = \boxed{3}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n + 3}$$
4. What is its explicit rule?

   $$u_n = \boxed{3n + 5}$$
5. Find the 50th term of the sequence.

   $$u_{50} = \boxed{155}$$

Answer:

1.
   - $u_1 - u_0 = 8 - 5 = 3$
   - $u_2 - u_1 = 11 - 8 = 3$
   - $u_3 - u_2 = 14 - 11 = 3$
2. Yes, the difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n + 3$
4. Explicit rule: $u_n = 3n + 5$
5. $u_{50} = 3 \times 50 + 5 = 155$

Ex 41: Consider the sequence $(4,\ 9,\ 14,\ 19, \ldots)$

1.
   - $u_1 - u_0 = \boxed{5}$
   - $u_2 - u_1 = \boxed{5}$
   - $u_3 - u_2 = \boxed{5}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n + 5}$$
4. What is its explicit rule?

   $$u_n = \boxed{5n + 4}$$
5. Find the 50th term of the sequence.

   $$u_{50} = \boxed{254}$$

Answer:

1.
   - $u_1 - u_0 = 9 - 4 = 5$
   - $u_2 - u_1 = 14 - 9 = 5$
   - $u_3 - u_2 = 19 - 14 = 5$
2. Yes, the difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n + 5$
4. Explicit rule: $u_n = 5n + 4$
5. $u_{50} = 5 \times 50 + 4 = 254$

Ex 42: Consider the sequence $(125,\ 115,\ 105,\ 95, \ldots)$

1.
   - $u_1 - u_0 = \boxed{-10}$
   - $u_2 - u_1 = \boxed{-10}$
   - $u_3 - u_2 = \boxed{-10}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n - 10}$$
4. What is its explicit rule?

   $$u_n = \boxed{-10n + 125}$$
5. Find the 1000th term of the sequence.

   $$u_{1000} = \boxed{-9875}$$

Answer:

1.
   - $u_1 - u_0 = 115 - 125 = -10$
   - $u_2 - u_1 = 105 - 115 = -10$
   - $u_3 - u_2 = 95 - 105 = -10$
2. Yes, the difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n - 10$
4. Explicit rule: $u_n = -10n + 125$
5. $u_{1000} = -10 \times 1000 + 125 = -9875$

D.2 DETERMINING THE EXPLICIT RULE OF AN ARITHMETIC SEQUENCE GIVEN TWO TERMS

Ex 43: For the arithmetic sequence given that $u_2 = 11$ and $u_6 = 31$, what is its explicit rule?

$$u_n = \boxed{5n+1}$$

Answer: The explicit rule for an arithmetic sequence is $u_n = u_0 + nd$. We need to solve the system of equations based on the given terms:

$$\begin{cases} u_0 + 2d = 11 & (1) \\ u_0 + 6d = 31 & (2) \end{cases}$$

Subtract equation (1) from equation (2):

$$(u_0 + 6d) - (u_0 + 2d) = 31 - 11$$

$$4d = 20$$

$$d = 5$$

Substitute $d = 5$ back into equation (1):

$$u_0 + 2 \times 5 = 11$$

$$u_0 + 10 = 11$$

$$u_0 = 1$$

Therefore, the explicit rule is $u_n = 5n + 1$.

Ex 44: For the arithmetic sequence given that $u_1 = 7$ and $u_{20} = 45$, what is its explicit rule?

$$u_n = \boxed{2n+5}$$

Answer: The explicit rule for an arithmetic sequence is $u_n = u_0 + nd$. We need to solve the system of equations based on the given terms:

$$\begin{cases} u_0 + d = 7 & (1) \\ u_0 + 20d = 45 & (2) \end{cases}$$

Subtract equation (1) from equation (2):

$$(u_0 + 20d) - (u_0 + d) = 45 - 7$$

$$19d = 38$$

$$d = 2$$

Substitute $d = 2$ back into equation (1):

$$u_0 + 2 = 7$$

$$u_0 = 5$$

Therefore, the explicit rule is $u_n = 2n + 5$.

Ex 45: For the arithmetic sequence given that $u_3 = 32$ and $u_{20} = 202$, what is its explicit rule?

$$u_n = \boxed{10n+2}$$

Answer: The explicit rule for an arithmetic sequence is $u_n = u_0 + nd$. We need to solve the system of equations based on the given terms:

$$\begin{cases} u_0 + 3d = 32 & (1) \\ u_0 + 20d = 202 & (2) \end{cases}$$

Subtract equation (1) from equation (2):

$$(u_0 + 20d) - (u_0 + 3d) = 202 - 32$$

$$17d = 170$$

$$d = 10$$

Substitute $d = 10$ back into equation (1):

$$u_0 + 3 \times 10 = 32$$

$$u_0 + 30 = 32$$

$$u_0 = 2$$

Therefore, the explicit rule is $u_n = 10n + 2$.

D.3 FINDING THE TERM NUMBER IN AN ARITHMETIC SEQUENCE

Ex 46: For an arithmetic sequence (u_n) with common difference $d = 2$ and $u_0 = 1$, find n for $u_n = 21$.

$$n = \boxed{10}$$

Answer: The general formula of an arithmetic sequence is

$$\begin{aligned} u_n &= u_0 + nd \\ &= 1 + 2n \quad (u_0 = 1,\, d = 2) \end{aligned}$$

Solve the equation $u_n = 21$:

$$\begin{aligned} u_n &= 21 \\ 1 + 2n &= 21 \\ 2n &= 20 \\ n &= 10. \end{aligned}$$

Ex 47: For an arithmetic sequence (u_n) with common difference $d = 4$ and $u_0 = -6$, find n for $u_n = 38$.

$$n = \boxed{11}$$

Answer: The general formula of an arithmetic sequence is

$$\begin{aligned} u_n &= u_0 + nd \\ &= -6 + 4n \quad (u_0 = -6,\, d = 4) \end{aligned}$$

Solve the equation $u_n = 38$:

$$\begin{aligned} u_n &= 38 \\ -6 + 4n &= 38 \\ 4n &= 38 + 6 \\ 4n &= 44 \\ n &= 11. \end{aligned}$$

Ex 48: For an arithmetic sequence (u_n) with common difference $d = -3$ and $u_0 = 20$, find n for $u_n = 2$.

$$n = \boxed{6}$$

Answer: The general formula of an arithmetic sequence is

$$\begin{aligned} u_n &= u_0 + nd \\ &= 20 - 3n \quad (u_0 = 20,\, d = -3) \end{aligned}$$

Solve the equation $u_n = 2$:

$$\begin{aligned} u_n &= 2 \\ 20 - 3n &= 2 \\ -3n &= 2 - 20 \\ -3n &= -18 \\ n &= 6. \end{aligned}$$

Ex 49: For an arithmetic sequence (u_n) with common difference $d = \frac{1}{2}$ and $u_0 = 2$, find n for $u_n = 6$.

$$n = \boxed{8}$$

Answer: The general formula of an arithmetic sequence is

$$\begin{aligned} u_n &= u_0 + nd \\ &= 2 + \frac{1}{2}n \quad (u_0 = 2,\, d = \frac{1}{2}) \end{aligned}$$

Solve the equation $u_n = 6$:

$$\begin{aligned} u_n &= 6 \\ 2 + \frac{1}{2}n &= 6 \\ \frac{1}{2}n &= 6 - 2 \\ \frac{1}{2}n &= 4 \\ n &= 8. \end{aligned}$$

D.4 MODELING REAL SITUATIONS

Ex 50: You start with \$30 and each week your parent gives you \$10.
Let u_n be the amount of money you have after n weeks.

1. Find the explicit formula for u_n.

$$u_n = \boxed{30 + 10n}$$

2. How much money will you have after 20 weeks?

$$\boxed{230} \text{ dollars}$$

Answer:

1. The explicit formula is:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr weeks} \times \text{Amount received each week} \\ &= 30 + n \times 10 \\ &= 30 + 10n \end{aligned}$$

2. After 20 weeks:

$$\begin{aligned} u_{20} &= 30 + 10 \times 20 \\ &= 30 + 200 \\ &= 230 \end{aligned}$$

So, after 20 weeks you will have \$230.

Ex 51: You deposit \$1 500 in a savings account that pays simple interest at a rate of 4% per year.
Let u_n be the amount of money in your account after n years.

1. Find the explicit formula for u_n.

$$u_n = \boxed{1500 + 60n}$$

2. What is your amount at year 20?

$$\boxed{2700} \text{ dollars}$$

Answer:

1. The explicit formula is:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr years} \times \text{Percentage of the initial an} \\ &= 1\,500 + n \times 0.04 \times 1\,500 \\ &= 1\,500 + 60n \end{aligned}$$

2. At year 20:

$$\begin{aligned} u_{20} &= 1\,500 + 60 \times 20 \\ &= 1\,500 + 1\,200 \\ &= 2\,700 \end{aligned}$$

So, your amount at year 20 is \$2 700.

Ex 52: You start a stamp collection with 12 stamps. Each month, you add 4 new stamps to your collection.
Let u_n be the number of stamps you have after n months.

1. Find the explicit formula for u_n.

$$u_n = \boxed{12 + 4n}$$

2. How many stamps will you have after 15 months?

$$\boxed{72} \text{ stamps}$$

Answer:

1. The explicit formula is:

$$\begin{aligned} u_n &= \text{Initial number of stamps} + \text{Nbr months} \times \text{Stamps added} \\ &= 12 + n \times 4 \\ &= 12 + 4n \end{aligned}$$

2. After 15 months:

$$\begin{aligned} u_{15} &= 12 + 4 \times 15 \\ &= 12 + 60 \\ &= 72 \end{aligned}$$

So, after 15 months you will have 72 stamps.

Ex 53: A school plants 5 trees in its garden to start. Every year, they plant 3 new trees.
Let u_n be the total number of trees after n years.

1. Find the explicit formula for u_n.

$$u_n = \boxed{5+3n}$$

2. How many trees will there be after 12 years?

$$\boxed{41} \text{ trees}$$

Answer:

1. The explicit formula is:

$$\begin{aligned} u_n &= \text{Initial number of trees} + \text{Nbr years} \times \text{Trees planted per year} \\ &= 5 + n \times 3 \\ &= 5 + 3n \end{aligned}$$

2. After 12 years:

$$\begin{aligned} u_{12} &= 5 + 3 \times 12 \\ &= 5 + 36 \\ &= 41 \end{aligned}$$

So, after 12 years there will be 41 trees.

E GEOMETRIC SEQUENCES

E.1 STUDYING A GEOMETRIC SEQUENCE

Ex 54: Consider the sequence $(3,\ 6,\ 12,\ 24,\ 48,\ \ldots)$

1.
   - $u_1 \div u_0 = \boxed{2}$
   - $u_2 \div u_1 = \boxed{2}$
   - $u_3 \div u_2 = \boxed{2}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant.
3. What is its recursive rule?
   $$u_{n+1} = \boxed{2u_n}$$
4. What is its explicit rule?
   $$u_n = \boxed{3 \times 2^n}$$
5. Find the 10th term of the sequence.
   $$u_{10} = \boxed{3072}$$

Answer:

1.
   - $u_1 \div u_0 = 6 \div 3 = 2$
   - $u_2 \div u_1 = 12 \div 6 = 2$
   - $u_3 \div u_2 = 24 \div 12 = 2$
2. Yes, the ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = 2u_n$
4. Explicit rule: $u_n = 3 \times 2^n$
5. $u_{10} = 3 \times 2^{10} = 3 \times 1024 = 3072$

Ex 55: Consider the sequence $(1,\ -1,\ 1,\ -1,\ 1,\ \ldots)$

1.
   - $u_1 \div u_0 = \boxed{-1}$
   - $u_2 \div u_1 = \boxed{-1}$
   - $u_3 \div u_2 = \boxed{-1}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant.
3. What is its recursive rule?
   $$u_{n+1} = \boxed{-1 \times u_n}$$
4. What is its explicit rule?
   $$u_n = \boxed{(-1)^n}$$
5. Find the 10th term of the sequence.
   $$u_{10} = \boxed{1}$$

Answer:

1.
   - $u_1 \div u_0 = -1 \div 1 = -1$
   - $u_2 \div u_1 = 1 \div (-1) = -1$
   - $u_3 \div u_2 = -1 \div 1 = -1$
2. Yes, the ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = -1 \times u_n$
4. Explicit rule: $u_n = (-1)^n$
5. $u_{10} = (-1)^{10} = 1$

Ex 56: Consider the sequence $(4,\ 2,\ 1,\ 0.5,\ 0.25,\ \ldots)$

1.
   - $u_1 \div u_0 = \boxed{0.5}$
   - $u_2 \div u_1 = \boxed{0.5}$
   - $u_3 \div u_2 = \boxed{0.5}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant.
3. What is its recursive rule?
   $$u_{n+1} = \boxed{0.5u_n}$$
4. What is its explicit rule?
   $$u_n = \boxed{4 \times (0.5)^n}$$
5. Find the 10th term of the sequence.
   $$u_{10} = \boxed{0.00390625}$$

Answer:

1.
   - $u_1 \div u_0 = 2 \div 4 = 0.5$
   - $u_2 \div u_1 = 1 \div 2 = 0.5$
   - $u_3 \div u_2 = 0.5 \div 1 = 0.5$
2. Yes, the ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = 0.5\,u_n$
4. Explicit rule: $u_n = 4 \times (0.5)^n$
5. $u_{10} = 4 \times (0.5)^{10} = 4 \times \frac{1}{1024} = 0.00390625$

E.2 FINDING THE TERM NUMBER IN AN GEOMETRIC SEQUENCE

Ex 57: For a geometric sequence (u_n) with common ratio $r = 2$ and $u_0 = 1$, find n for $u_n = 8$.

$$n = \boxed{3}$$

Answer: The general formula of a geometric sequence is

$$\begin{aligned} u_n &= u_0 \times r^n \\ &= 2^n \qquad (u_0 = 1,\ r = 2) \end{aligned}$$

Solve the equation $2^n = 8$:

- **Trial and error Method:**
  - for $n = 1$, $2^1 = 2$
  - for $n = 2$, $2^2 = 4$
  - for $n = 3$, $2^3 = 8$

  So $n = 3$.
- **Analytical Method:**

$$\begin{aligned} 2^n &= 8 \\ \ln(2^n) &= \ln(8) \\ n\ln(2) &= \ln(8) \\ n &= \frac{\ln(8)}{\ln(2)} \\ n &= 3. \end{aligned}$$

  So $n = 3$.

Ex 58: For a geometric sequence (u_n) with common ratio $r = 3$ and $u_0 = 1$, find n for $u_n = 81$.

$$n = \boxed{4}$$

Answer: The general formula of a geometric sequence is

$$\begin{aligned} u_n &= u_0 \times r^n \\ &= 3^n \qquad (u_0 = 1,\ r = 3) \end{aligned}$$

Solve the equation $3^n = 81$:

- **Trial and Error Method:**
  - for $n = 1$, $3^1 = 3$
  - for $n = 2$, $3^2 = 9$
  - for $n = 3$, $3^3 = 27$
  - for $n = 4$, $3^4 = 81$

  So $n = 4$.
- **Analytical Method:**

$$\begin{aligned} 3^n &= 81 \\ \ln(3^n) &= \ln(81) \\ n\ln(3) &= \ln(81) \\ n &= \frac{\ln(81)}{\ln(3)} \\ n &= 4. \end{aligned}$$

  So $n = 4$.

Ex 59: For a geometric sequence (u_n) with common ratio $r = 10$ and $u_0 = 2$, find n for $u_n = 2000$.

$$n = \boxed{3}$$

Answer: The general formula of a geometric sequence is

$$\begin{aligned} u_n &= u_0 \times r^n \\ &= 2 \times 10^n \quad (u_0 = 2,\ r = 10) \end{aligned}$$

Solve the equation $2 \times 10^n = 2000$:

- **Trial and Error Method:**
  - for $n = 1$, $2 \times 10^1 = 20$
  - for $n = 2$, $2 \times 10^2 = 200$
  - for $n = 3$, $2 \times 10^3 = 2000$

  So $n = 3$.
- **Analytical Method:**

$$\begin{aligned} 2 \times 10^n &= 2000 \\ 10^n &= 1000 \\ n\ln(10) &= \ln(1000) \\ n &= \frac{\ln(1000)}{\ln(10)} \\ n &= 3 \end{aligned}$$

  So $n = 3$.

Ex 60: For a geometric sequence (u_n) with common ratio $r = 2$ and $u_0 = 10$, find n for $u_n = 160$.

$$n = \boxed{4}$$

Answer: The general formula of a geometric sequence is

$$\begin{aligned} u_n &= u_0 \times r^n \\ &= 10 \times 2^n \quad (u_0 = 10,\ r = 2) \end{aligned}$$

Solve the equation $10 \times 2^n = 160$:

- **Trial and error Method:**
  - for $n = 1$, $10 \times 2^1 = 20$
  - for $n = 2$, $10 \times 2^2 = 40$
  - for $n = 3$, $10 \times 2^3 = 80$
  - for $n = 4$, $10 \times 2^4 = 160$

  So $n = 4$.
- **Analytical Method:**

$$\begin{aligned} 10 \times 2^n &= 160 \\ 2^n &= 16 \\ n &= \frac{\ln(16)}{\ln(2)} \\ n &= \frac{\ln(2^4)}{\ln(2)} \\ n &= \frac{4\ln(2)}{\ln(2)} \\ n &= 4 \end{aligned}$$

  So $n = 4$.

E.3 MODELING REAL SITUATIONS

Ex 61: A scientist observes a culture of bacteria. At the start, there are $u_0 = 5$ bacteria in a petri dish. Each day, the number of bacteria doubles.
Let u_n be the number of bacteria at the day n.

1. What are the first three terms of the sequence (u_n)?
   - $u_1 = \boxed{10}$ bacteria
   - $u_2 = \boxed{20}$ bacteria
   - $u_3 = \boxed{40}$ bacteria
2. What is its recursive rule?
$$u_{n+1} = \boxed{2 \times u_n}$$
3. What is its explicit rule?
$$u_n = \boxed{5 \times 2^n}$$
4. How many bacteria will there be after 10 days?
$$u_{10} = \boxed{5120} \text{ bacteria}$$

Answer: The number of bacteria doubles each day:

1. - $u_1 = 2 \times u_0 = 2 \times 5 = 10$
   - $u_2 = 2 \times u_1 = 2 \times 10 = 20$
   - $u_3 = 2 \times u_2 = 2 \times 20 = 40$
2. The rule is multiply by 2:
$$u_{n+1} = 2 \times u_n$$
3. The explicit rule is obtained by iterating the recursive relation:
$$u_n = 5 \times 2^n$$
4. After 10 days:
$$u_{10} = 5 \times 2^{10} = 5 \times 1024 = 5120$$

Ex 62: An investor starts with \$100 in a high-risk fund. Each year, the value of the investment triples.
Let u_n be the value of the investment after n years.

1. What are the first three terms of the sequence (u_n)?
   - $u_1 = \boxed{300}$ dollars
   - $u_2 = \boxed{900}$ dollars
   - $u_3 = \boxed{2700}$ dollars
2. What is its recursive rule?
$$u_{n+1} = \boxed{3 \times u_n}$$
3. What is its explicit rule?
$$u_n = \boxed{100 \times 3^n}$$
4. How many dollars will the investment be worth after 10 years?
$$u_{10} = \boxed{5904900} \text{ dollars}$$

Answer: The value of the investment triples each year:

1. - $u_1 = 3 \times u_0 = 3 \times 100 = 300$
   - $u_2 = 3 \times u_1 = 3 \times 300 = 900$
   - $u_3 = 3 \times u_2 = 3 \times 900 = 2700$
2. The rule is multiply by 3:
$$u_{n+1} = 3 \times u_n$$
3. The explicit rule is obtained by iterating the recursive relation:
$$u_n = 100 \times 3^n$$
4. After 10 years:
$$u_{10} = 100 \times 3^{10} = 100 \times 59049 = 5904900$$

Ex 63: A computer virus infects 4 computers initially. Each day, the number of infected computers quadruples.
Let u_n be the number of infected computers at day n.

1. What are the first three terms of the sequence (u_n)?
   - $u_1 = \boxed{16}$ computers
   - $u_2 = \boxed{64}$ computers
   - $u_3 = \boxed{256}$ computers
2. What is its recursive rule?
$$u_{n+1} = \boxed{4 \times u_n}$$
3. What is its explicit rule?
$$u_n = \boxed{4 \times 4^n}$$
4. How many computers will be infected after 10 days?
$$u_{10} = \boxed{4194304} \text{ computers}$$

Answer: The number of infected computers quadruples each day:

1. - $u_1 = 4 \times u_0 = 4 \times 4 = 16$
   - $u_2 = 4 \times u_1 = 4 \times 16 = 64$
   - $u_3 = 4 \times u_2 = 4 \times 64 = 256$
2. The rule is multiply by 4:
$$u_{n+1} = 4 \times u_n$$
3. The explicit rule is obtained by iterating the recursive relation:
$$u_n = 4 \times 4^n$$
4. After 10 days:
$$u_{10} = 4 \times 4^{10} = 4 \times 1048576 = 4194304$$

Ex 64: You deposit \$1000 in a savings account that earns 5% compound interest per year. Compound interest means that each year, interest is calculated on the current balance (principal plus any previously earned interest), and added to the balance.
Let u_n be the amount in the account after n years.

1. What are the first three terms of the sequence (u_n)?
   - $u_1 = \boxed{1050}$ dollars
   - $u_2 = \boxed{1102.5}$ dollars
   - $u_3 = \boxed{1157.625}$ dollars
2. What is its recursive rule?
$$u_{n+1} = \boxed{1.05 \times u_n}$$
3. What is its explicit rule?
$$u_n = \boxed{1000 \times 1.05^n}$$
4. How many dollars will be in the account after 10 years? (Round to two decimal places if necessary.)
$$u_{10} = \boxed{1628.89} \text{ dollars}$$

Answer: The amount in the account increases by 5% each year on the current balance:

1. - $u_1 = 1.05 \times u_0 = 1.05 \times 1000 = 1050$
   - $u_2 = 1.05 \times u_1 = 1.05 \times 1050 = 1102.5$
   - $u_3 = 1.05 \times u_2 = 1.05 \times 1102.5 = 1157.625$
2. The rule is multiply by 1.05:
$$u_{n+1} = 1.05 \times u_n$$
3. The explicit rule is obtained by iterating the recursive relation:
$$u_n = 1000 \times 1.05^n$$
4. After 10 years:
$$u_{10} = 1000 \times 1.05^{10} \approx 1628.89$$

Ex 65: You buy a car for \$10,000. It depreciates at a rate of 20% per year. Depreciation means that each year, the value decreases by 20% of the current value, so the remaining value is 80% of the previous year's value. Thus, the value multiplies by 0.8 each year.
Let u_n be the value of the car after n years. What are the first three terms of the sequence (u_n)?

1. - $u_1 = \boxed{8000}$ dollars
   - $u_2 = \boxed{6400}$ dollars
   - $u_3 = \boxed{5120}$ dollars
2. What is its recursive rule?
$$u_{n+1} = \boxed{0.8 \times u_n}$$
3. What is its explicit rule?
$$u_n = \boxed{10000 \times 0.8^n}$$
4. How many dollars will the car be worth after 10 years? (Round to two decimal places if necessary.)
$$u_{10} = \boxed{1073.74} \text{ dollars}$$

Answer: The value of the car decreases by 20% each year, multiplying by 0.8:

1. - $u_1 = 0.8 \times u_0 = 0.8 \times 10000 = 8000$
   - $u_2 = 0.8 \times u_1 = 0.8 \times 8000 = 6400$
   - $u_3 = 0.8 \times u_2 = 0.8 \times 6400 = 5120$
2. The rule is multiply by 0.8:
$$u_{n+1} = 0.8 \times u_n$$
3. The explicit rule is obtained by iterating the recursive relation:
$$u_n = 10000 \times 0.8^n$$
4. After 10 years:
$$u_{10} = 10000 \times 0.8^{10} \approx 1073.74$$

F SERIES

F.1 COMPUTING TERMS AND SUMS OF SEQUENCES: LEVEL 1

Ex 66: Consider the sequence, (u_n), defined by $(2, 5, 8, 11, \dots)$. Find

1. $u_0 = \boxed{2}$
2. $u_1 = \boxed{5}$
3. $u_2 = \boxed{8}$
4. $S_0 = \boxed{2}$
5. $S_1 = \boxed{7}$
6. $S_2 = \boxed{15}$

Answer:

1. $u_0 = 2$
2. $u_1 = 5$
3. $u_2 = 8$
4. $S_0 = u_0$
   $= 2$
5. $S_1 = u_0 + u_1$
   $= 2 + 5$
   $= 7$
6. $S_2 = u_0 + u_1 + u_2$
   $= 2 + 5 + 8$
   $= 15$

Ex 67: Consider the sequence (u_n) defined by $(2, 4, 8, 16, 32, \ldots)$. Find:

1. $u_0 = \boxed{2}$
2. $u_1 = \boxed{4}$
3. $u_2 = \boxed{8}$
4. $S_0 = \boxed{2}$
5. $S_1 = \boxed{6}$
6. $S_2 = \boxed{14}$

Answer:

1. $u_0 = 2$
2. $u_1 = 4$
3. $u_2 = 8$
4. $S_0 = u_0$
   $= 2$
5. $S_1 = u_0 + u_1$
   $= 2 + 4$
   $= 6$
6. $S_2 = u_0 + u_1 + u_2$
   $= 2 + 4 + 8$
   $= 14$

Ex 68: Consider the sequence (u_n) defined by $(5, -5, -15, -25, \ldots)$. Find:

1. $u_0 = \boxed{5}$
2. $u_1 = \boxed{-5}$
3. $u_2 = \boxed{-15}$
4. $S_0 = \boxed{5}$
5. $S_1 = \boxed{0}$
6. $S_2 = \boxed{-15}$

Answer:

1. $u_0 = 5$
2. $u_1 = -5$
3. $u_2 = -15$
4. $S_0 = u_0$
   $= 5$
5. $S_1 = u_0 + u_1$
   $= 5 + (-5)$
   $= 0$
6. $S_2 = u_0 + u_1 + u_2$
   $= 5 + (-5) + (-15)$
   $= 0 + (-15)$
   $= -15$

F.2 COMPUTING TERMS AND SUMS OF SEQUENCES: LEVEL 2

Ex 69: For the sequence (u_n) defined by $u_n = n^2$, find:

$$S_3 = \boxed{14}$$

Answer:

- $u_0 = 0^2 = 0$
- $u_1 = 1^2 = 1$
- $u_2 = 2^2 = 4$
- $u_3 = 3^2 = 9$
- $S_3 = u_0 + u_1 + u_2 + u_3$
  $= 0 + 1 + 4 + 9$
  $= 14$

Ex 70: For the sequence (u_n) defined by $u_0 = 2$ and $u_{n+1} = 2u_n - 1$, find:

$$S_3 = \boxed{19}$$

Answer: First, compute the terms:

- $u_0 = 2$
- $u_1 = 2u_0 - 1 = 2 \times 2 - 1 = 3$
- $u_2 = 2u_1 - 1 = 2 \times 3 - 1 = 5$
- $u_3 = 2u_2 - 1 = 2 \times 5 - 1 = 9$
- $S_3 = u_0 + u_1 + u_2 + u_3$
  $= 2 + 3 + 5 + 9$
  $= 19$

Ex 71: For the sequence (u_n) defined by $u_n = n^2 - 5n$, find:

$$S_3 = \boxed{-16}$$

Answer: Calculate the first terms:

- $u_0 = 0^2 - 5 \times 0 = 0$
- $u_1 = 1^2 - 5 \times 1 = 1 - 5 = -4$
- $u_2 = 2^2 - 5 \times 2 = 4 - 10 = -6$
- $u_3 = 3^2 - 5 \times 3 = 9 - 15 = -6$
- $S_3 = u_0 + u_1 + u_2 + u_3$
  $= 0 + (-4) + (-6) + (-6)$
  $= -16$

Ex 72: For the sequence (u_n) defined by $u_{n+1} = u_n + n$ with $u_0 = 2$, find:

$$S_3 = \boxed{12}$$

Answer:

- $u_0 = 2$
- $u_1 = u_0 + 0 = 2 + 0 = 2$
- $u_2 = u_1 + 1 = 2 + 1 = 3$
- $u_3 = u_2 + 2 = 3 + 2 = 5$
- $S_3 = u_0 + u_1 + u_2 + u_3$
  $= 2 + 2 + 3 + 5$
  $= 12$

G SUM OF AN ARITHMETIC SEQUENCE

G.1 CALCULATING SUMS OF ARITHMETIC SEQUENCES: LEVEL 1

Ex 73: Calculate the sum

$$1+2+3+4+5+6+7+8+9+10+11 = \boxed{66}$$

Answer:

- The sequence $(1,2,3,4,5,6,7,8,9,10,11,\dots)$ is an arithmetic sequence with common difference $d=1$, first term $u_0=1$, and the 11th term $u_{10}=11$.
- Using the formula for the sum of the first $(n+1)$ terms of an arithmetic sequence$S_n = \frac{n+1}{2}(u_0+u_n)$,we have

$$\begin{aligned} S_{10} &= 1+2+3+4+5+6+7+8+9+10+11 \\ &= \frac{10+1}{2}(1+11) \\ &= \frac{11}{2}\times 12 \\ &= 66 \end{aligned}$$

Ex 74: Calculate the sum

$$5+10+15+20+25+30+35+40+45+50 = \boxed{275}$$

Answer:

- The sequence $(5,10,15,20,25,30,35,40,45,50,\dots)$ is an arithmetic sequence with common difference $d=5$, first term $u_0=5$, and the 10th term $u_9=50$.
- Using the formula for the sum of the first $(n+1)$ terms of an arithmetic sequence$S_n = \frac{n+1}{2}(u_0+u_n)$,we have

$$\begin{aligned} S_9 &= 5+10+15+20+25+30+35+40+45+50 \\ &= \frac{9+1}{2}(5+50) \\ &= \frac{10}{2}\times 55 \\ &= 275 \end{aligned}$$

Ex 75: Calculate the sum

$$5+11+17+23+29+35+41+47 = \boxed{208}$$

Answer:

- The sequence $(5,11,17,23,29,35,41,47,\dots)$ is an arithmetic sequence with common difference $d=6$, first term $u_0=5$, and the 8th term $u_7=47$.
- Using the formula for the sum of the first $(n+1)$ terms of an arithmetic sequence$S_n = \frac{n+1}{2}(u_0+u_n)$,we have

$$\begin{aligned} S_7 &= 5+11+17+23+29+35+41+47 \\ &= \frac{7+1}{2}(5+47) \\ &= \frac{8}{2}\times 52 \\ &= 208 \end{aligned}$$

G.2 CALCULATING SUMS OF ARITHMETIC SEQUENCES: LEVEL 2

Ex 76: Calculate the sum

$$3+7+11+15+\dots+103 = \boxed{1378}$$

Answer:

- The sequence $(3,7,11,15,\dots)$ is an arithmetic sequence with common difference $d=4$, first term $u_0=3$.
- We need to find n such that $u_n=103$.The general formula of an arithmetic sequence is

$$\begin{aligned} u_n &= u_0+nd \\ &= 3+4n \qquad (u_0=3,\ d=4) \end{aligned}$$

Solve the equation $u_n=103$:

$$\begin{aligned} u_n &= 103 \\ 3+4n &= 103 \\ 4n &= 100 \\ n &= 25. \end{aligned}$$

- Using the formula for the sum of the first $(n+1)$ terms of an arithmetic sequence$S_n = \frac{n+1}{2}(u_0+u_n)$,we have

$$\begin{aligned} S_{25} &= 3+7+11+15+\dots+103 \\ &= \frac{25+1}{2}(3+103) \\ &= \frac{26}{2}\times 106 \\ &= 1378 \end{aligned}$$

Ex 77: Calculate the sum

$$6+9+12+\dots+48 = \boxed{324}$$

Answer:

- The sequence $(6,9,12,\dots)$ is an arithmetic sequence with common difference $d=3$, first term $u_0=6$.
- We need to find n such that $u_n=48$.

$$\begin{aligned} u_n &= u_0+nd \\ &= 6+3n \\ 6+3n &= 48 \\ 3n &= 42 \\ n &= 14 \end{aligned}$$

- Using the formula $S_n = \frac{n+1}{2}(u_0+u_n)$,

$$\begin{aligned} S_{14} &= 6+9+12+\dots+48 \\ &= \frac{14+1}{2}(6+48) \\ &= \frac{15}{2}\times 54 \\ &= 405 \end{aligned}$$

Ex 78: Calculate the sum

$$20+17+14+\cdots+(-1)=\boxed{110}$$

Answer:

- The sequence $(20, 17, 14, \dots)$ is an arithmetic sequence with common difference $d=-3$, first term $u_0=20$.
- We need to find n such that $u_n=-1$.

$$\begin{aligned} u_n &= u_0+nd \\ &= 20+n\times(-3) \\ &= 20-3n \\ 20-3n &= -1 \\ -3n &= -21 \\ n &= 7 \end{aligned}$$

- Using the formula $S_n=\frac{n+1}{2}(u_0+u_n)$,

$$\begin{aligned} S_7 &= 20+17+14+\cdots+(-1) \\ &= \frac{7+1}{2}(20+(-1)) \\ &= 4\times 19 \\ &= 76 \end{aligned}$$

Ex 79: Calculate the sum

$$4+8+12+\cdots+40=\boxed{220}$$

Answer:

- The sequence $(4, 8, 12, \dots)$ is an arithmetic sequence with common difference $d=4$, first term $u_0=4$.
- We need to find n such that $u_n=40$.

$$\begin{aligned} u_n &= u_0+nd \\ &= 4+4n \\ 4+4n &= 40 \\ 4n &= 36 \\ n &= 9 \end{aligned}$$

- Using the formula $S_n=\frac{n+1}{2}(u_0+u_n)$,

$$\begin{aligned} S_9 &= 4+8+12+\cdots+40 \\ &= \frac{9+1}{2}(4+40) \\ &= 5\times 44 \\ &= 220 \end{aligned}$$

H SUM OF AN GEOMETRIC SEQUENCE

H.1 CALCULATING SUMS OF GEOMETRIC SEQUENCES BY DIRECT ADDITION

Ex 80: Calculate the sum by adding the terms directly

$$1+2+4+8+16=\boxed{31}$$

Answer: By direct addition: $1+2+4+8+16=31$.

Ex 81: Calculate the sum by adding the terms directly

$$3^0+3^1+3^2+3^3+3^4+3^5=\boxed{364}$$

Answer: Using a calculator

$$\begin{aligned} 3^0+3^1+3^2+3^3+3^4+3^5 &= 1+3+9+27+81+243 \\ &= 364 \end{aligned}$$

Ex 82: Calculate the sum by adding the terms directly

$$2^0+2^1+2^2+2^3+2^4+2^5=\boxed{63}$$

Answer: Using a calculator

$$\begin{aligned} 2^0+2^1+2^2+2^3+2^4+2^5 &= 1+2+4+8+16+32 \\ &= 63 \end{aligned}$$

Ex 83: Calculate the sum by adding the terms directly

$$2\times10^0+2\times10^1+2\times10^2+2\times10^3+2\times10^4=\boxed{22222}$$

Answer: Using a calculator:

$$\begin{aligned} &2\times10^0+2\times10^1+2\times10^2+2\times10^3+2\times10^4 \\ &= 2\times1+2\times10+2\times100+2\times1000+2\times10000 \\ &= 2+20+200+2000+20000 \\ &= 22222 \end{aligned}$$

H.2 CALCULATING SUMS OF GEOMETRIC SEQUENCES: LEVEL 1

Ex 84: Calculate the sum

$$3^0+3^1+3^2+3^3+3^4+\cdots+3^{10}=\boxed{88572}$$

Answer: Using the geometric series formula $S_n=u_0\cdot\frac{1-r^{n+1}}{1-r}$ with $n=10$, $u_0=1$ and $r=3$,

$$\begin{aligned} S_{10} &= 3^0+3^1+3^2+\cdots+3^{10} \\ &= 1\cdot\frac{1-3^{11}}{1-3} \\ &= 88573 \end{aligned}$$

Ex 85: Calculate the sum

$$2^0+2^1+2^2+2^3+2^4+\cdots+2^9=\boxed{1023}$$

Answer: Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 9$, $u_0 = 1$, $r = 2$,

$$\begin{aligned} S_9 &= 2^0 + 2^1 + 2^2 + \cdots + 2^9 \\ &= 1 \cdot \frac{1-2^{10}}{1-2} \\ &= \frac{1-1024}{-1} \\ &= 1023 \end{aligned}$$

Ex 86: Calculate the sum

$$3 \times 5^0 + 3 \times 5^1 + 3 \times 5^2 + \cdots + 3 \times 5^6 + 3 \times 5^7 = \boxed{292968}$$

Answer: Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 7$, $u_0 = 3$, $r = 5$,

$$\begin{aligned} S_7 &= 3 \times 5^0 + 3 \times 5^1 + 3 \times 5^2 + \cdots + 3 \times 5^7 \\ &= 3 \cdot \frac{1-5^8}{1-5} \\ &= 3 \cdot \frac{1-390625}{-4} \\ &= 3 \cdot \frac{-390624}{-4} \\ &= 3 \times 97656 \\ &= 292968 \end{aligned}$$

Ex 87: Calculate the sum

$$5 \times 4^0 + 5 \times 4^1 + 5 \times 4^2 + \cdots + 5 \times 4^8 = \boxed{436905}$$

Answer: Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 8$, $u_0 = 5$, $r = 4$,

$$\begin{aligned} S_8 &= 5 \times 4^0 + 5 \times 4^1 + 5 \times 4^2 + \cdots + 5 \times 4^8 \\ &= 5 \cdot \frac{1-4^9}{1-4} \\ &= 5 \cdot \frac{1-262144}{-3} \\ &= 5 \cdot \frac{-262143}{-3} \\ &= 5 \times 87381 \\ &= 436905 \end{aligned}$$

H.3 CALCULATING SUMS OF GEOMETRIC SEQUENCES: LEVEL 2

Ex 88: Calculate the sum

$$1 + 2 + 4 + 8 + \cdots + 256 = \boxed{511}$$

Answer:

- The sequence $(1, 2, 4, 8, \dots)$ is a geometric sequence with first term $u_0 = 1$ and common ratio $r = 2$.
- To find n such that $u_n = 256$, solve the equation $u_n = 256$:

$$\begin{aligned} u_n &= u_0 \cdot r^n \\ 256 &= 1 \cdot 2^n \\ 2^n &= 256 \\ 2^n &= 2^8 \\ n &= 8 \end{aligned}$$

Alternatively, using logarithms:

$$\begin{aligned} 2^n &= 256 \\ \ln(2^n) &= \ln(256) \\ n\ln(2) &= \ln(256) \\ n &= \frac{\ln(256)}{\ln(2)} \\ n &= 8 \end{aligned}$$

So, the sequence has $n + 1 = 9$ terms (from $n = 0$ to $n = 8$).

- Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 8$, $u_0 = 1$, $r = 2$:

$$\begin{aligned} S_8 &= 1 + 2 + 4 + 8 + \cdots + 256 \\ &= 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^8 \\ &= 1 \cdot \frac{1-2^9}{1-2} \\ &= \frac{1-2^9}{-1} \\ &= \frac{1-512}{-1} \\ &= \frac{-511}{-1} \\ &= 511 \end{aligned}$$

Ex 89: Calculate the sum

$$2 + 6 + 18 + 54 + 162 + 486 + 1458 = \boxed{2186}$$

Answer:

- The sequence $(2,\ 6,\ 18,\ 54, \dots)$ is a geometric sequence with first term $u_0 = 2$ and common ratio $r = 3$.
- To find n such that $u_n = 1458$, solve the equation $u_n = 1458$:

$$\begin{aligned} u_n &= u_0 \cdot r^n \\ 1458 &= 2 \cdot 3^n \\ 3^n &= \frac{1458}{2} = 729 \\ 3^n &= 3^6 \\ n &= 6 \end{aligned}$$

Or with logarithms:

$$\begin{aligned} 3^n &= 729 \\ \ln(3^n) &= \ln(729) \\ n\ln(3) &= \ln(729) \\ n &= \frac{\ln(729)}{\ln(3)} \\ n &= 6 \end{aligned}$$

So, the sequence has $n + 1 = 7$ terms (from $n = 0$ to $n = 6$).

- Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 6$, $u_0 = 2$, $r = 3$:

$$\begin{aligned}
S_6 &= 2 + 6 + \cdots + 1458 \\
&= 2 \cdot 3^0 + 2 \cdot 3^1 + \cdots + 2 \cdot 3^6 \\
&= 2 \cdot \frac{1 - 3^7}{1 - 3} \\
&= 2 \cdot \frac{1 - 2187}{-2} \\
&= 2 \cdot \frac{-2186}{-2} \\
&= 2 \cdot 1093 \\
&= 2186
\end{aligned}$$

Ex 90: Calculate the sum

$$3 + 15 + \cdots + 1171875 = \boxed{1464843}$$

Answer:

- The sequence (3, 15, 75, 375, ...) is a geometric sequence with first term $u_0 = 3$ and common ratio $r = 5$.
- To find n such that $u_n = 1171875$, solve the equation $u_n = 1171875$:

$$\begin{aligned}
u_n &= u_0 \cdot r^n \\
1171875 &= 3 \cdot 5^n \\
5^n &= \frac{1171875}{3} = 390625 \\
5^n &= 5^8 \\
n &= 8
\end{aligned}$$

  Or with logarithms:

$$\begin{aligned}
5^n &= 390625 \\
\ln(5^n) &= \ln(390625) \\
n \ln(5) &= \ln(390625) \\
n &= \frac{\ln(390625)}{\ln(5)} \\
n &= 8
\end{aligned}$$

  So, the sequence has $n + 1 = 9$ terms (from $n = 0$ to $n = 8$).

- Using the geometric series formula $S_n = u_0 \cdot \frac{1-r^{n+1}}{1-r}$ with $n = 8$, $u_0 = 3$, $r = 5$:

$$\begin{aligned}
S_8 &= 3 + 15 + \cdots + 1171875 \\
&= 3 \cdot 5^0 + 3 \cdot 5^1 + \cdots + 3 \cdot 5^8 \\
&= 3 \cdot \frac{1 - 5^9}{1 - 5} \\
&= 3 \cdot \frac{1 - 1953125}{-4} \\
&= 3 \cdot \frac{-1953124}{-4} \\
&= 3 \times 488281 \\
&= 1464843
\end{aligned}$$