SEQUENCES

A NUMERICAL SEQUENCE

Definition Numerical Sequence

A **numerical sequence**, (u_n) is an ordered list of numbers $(u_0, u_1, u_2, \ldots)$ defined by a rule.
The number u_n is called the **nth term** of the sequence.

Ex: What is u_4 of this sequence?

n	0	1	2	3	4	5	...
u_n	3	5	7	9	11	13	...

Answer: $u_4 = 11$.

B DEFINITION USING A RECURSIVE RULE

Discover: Let's consider a sequence where the first term is 2, and each term is obtained by adding 3 to the previous term. The terms are:

$$(\underset{u_0}{2} \overset{+3}{\curvearrowright} \underset{u_1}{5} \overset{+3}{\curvearrowright} \underset{u_2}{8} \overset{+3}{\curvearrowright} \underset{u_3}{11}, \ldots)$$

We observe:

- $5 = 2+3 \longrightarrow u_1 = u_0+3 \longrightarrow u_{0+1} = u_0 + 3$
- $8 = 5+3 \longrightarrow u_2 = u_1+3 \longrightarrow u_{1+1} = u_1 + 3$
- $11 = 8+3 \longrightarrow u_3 = u_2+3 \longrightarrow u_{2+1} = u_2 + 3$
- $\vdots$
- So the rule is $u_{n+1} = u_n + 3$

$$(u_0 \overset{+3}{\curvearrowright} u_1 \overset{+3}{\curvearrowright} u_2 \overset{+3}{\curvearrowright} u_3, \ldots, u_n \overset{+3}{\curvearrowright} u_{n+1}, \ldots)$$

Definition Recursive Rule

A sequence can be defined by:

- the **first term** (starting number): u_0
- a **recursive rule** that tells how to obtain each term from the previous one:

$$u_{n+1} = \text{expression in } u_n$$

Ex: Write the recursive rule when each term is obtained by adding 2 to the previous term.

Answer:

$$u_{n+1} = u_n + 2$$

C DEFINITION USING AN EXPLICIT RULE

Definition Explicit Rule

A sequence can also be defined by an **explicit rule** (or **explicit formula**), which gives a direct formula for the nth term in terms of n:

$$u_n = \text{expression in } n$$

Ex: Consider the sequence defined by the explicit formula: $u_n = 3n + 2$.
Write the first five terms of this sequence.

Answer:

- For $n = 0$:

$$\begin{aligned} u_0 &= 3 \times 0 + 2 \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

- For $n = 1$:

$$\begin{aligned} u_1 &= 3 \times 1 + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

- For $n = 2$:

$$\begin{aligned} u_2 &= 3 \times 2 + 2 \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

- For $n = 3$:

$$\begin{aligned} u_3 &= 3 \times 3 + 2 \\ &= 9 + 2 \\ &= 11 \end{aligned}$$

- For $n = 4$:

$$\begin{aligned} u_4 &= 3 \times 4 + 2 \\ &= 12 + 2 \\ &= 14 \end{aligned}$$

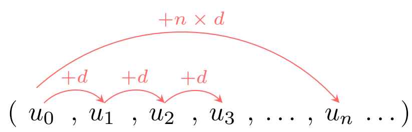
So the first five terms are: $2,\ 5,\ 8,\ 11,\ 14$.

D ARITHMETIC SEQUENCES

Definition Arithmetic Sequence

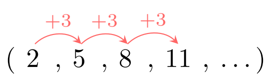
An **arithmetic sequence** is a sequence where the difference between consecutive terms is constant. This constant is called the **common difference** and is denoted by d.

- The recursive rule is: $u_{n+1} = u_n + d$
- The explicit formula is: $u_n = u_0 + nd$



Ex: Determine if the sequence $(2, 5, 8, 11, 14, \ldots)$ is arithmetic and find the common difference d if it is.

Answer:



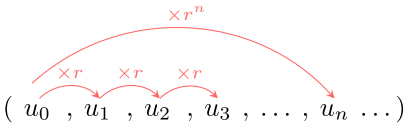
The differences between consecutive terms are: $5 - 2 = 3$, $8 - 5 = 3$, $11 - 8 = 3$, $14 - 11 = 3$.
Since the difference is constant and equal to 3, the sequence is arithmetic with $d = 3$.

E GEOMETRIC SEQUENCES

Definition Geometric Sequence

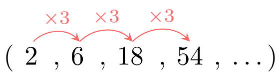
An **geometric sequence** is a sequence where the ratio between consecutive terms is constant. This constant is called the **common ratio** and is denoted by r.

- The recursive rule is: $u_{n+1} = u_n \times r$
- The explicit formula is: $u_n = u_0 \times r^n$



Ex: Determine if the sequence $(2, 6, 18, 54, 162, \dots)$ is geometric and find the common ratio r if it is.

Answer:



The ratios between consecutive terms are:$6 \div 2 = 3$, $18 \div 6 = 3$, $54 \div 18 = 3$, $162 \div 54 = 3$.
Since the ratio is constant and equal to 3, the sequence is geometric with $r = 3$.

F SERIES

Definition Series

A **series** is the sum of the terms of a sequence.

$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + \ldots + u_n \\ &= \sum_{i=0}^{n} u_i \end{aligned}$$

G SUM OF AN ARITHMETIC SEQUENCE

Discover: We want to calculate

$$S_{19} = \overbrace{5 + 10 + 15 + \ldots + 90 + 95 + 100}^{20 \text{ terms}}$$

The first term is $u_0 = 5$ and the last term is $u_{19} = 100$. However, we can also write:

$$\begin{array}{rrrrrrrrrr} & S_{19} = & & 5+ & 10+ & 15+ & \ldots+ & 90+ & 95+ & 100 & \\ + & S_{19} = & & 100+ & 95+ & 90+ & \ldots+ & 15+ & 10+ & 5 & \text{(reversing the terms)} \\ \hline & 2S_{19} = & & 105+ & 105+ & 105+ & \ldots+ & 105+ & 105+ & 105 & \text{(adding)} \\ & 2S_{19} = & 20 \times 105 & & & & & & & & \text{(20 times the same term)} \end{array}$$

So

$$S_{19} = \frac{20}{2} \times 105 = 1050$$

Proposition Sum of an Arithmetic Sequence

The sum of an arithmetic sequence is

$$S_n = \frac{n+1}{2}(u_0 + u_n)$$

H SUM OF AN GEOMETRIC SEQUENCE

Proposition Sum of a Geometric Sequence

The sum of a geometric sequence is

$$S_n = u_0 \cdot \frac{1 - r^{n+1}}{1 - r}$$

where r is the common ratio.