SEQUENCES

A NUMERICAL SEQUENCE

A.1 FINDING u_n

Ex 1:

n	1	2	3	4	5	6
u_n	3	5	7	9	11	13

What is u_4?

9

Answer: $u_4 = 9$.

Ex 2:

n	1	2	3	4	5	6
u_n	2	6	12	20	30	42

What is u_5?

30

Answer: $u_5 = 30$.

Ex 3:

n	1	2	3	4	5	6	7	8
u_n	4	9	16	25	36	49	64	81

What is u_7?

64

Answer: $u_7 = 64$.

Ex 4:

n	1	2	3	4	5	6	7	8
u_n	1	3	7	15	31	63	127	255

What is u_8?

255

Answer: $u_8 = 255$.

A.2 FINDING u_n IN AN ARITHMETIC SEQUENCE

Ex 5: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	3	5	7	9	11	13

Answer: $u_6 = 13$, because each term increases by 2.

n	1	2	3	4	5	6
u_n	3	5	7	9	11	13

+2 +2 +2 +2 +2

Ex 6: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	3	8	13	18	23	28

Answer: $u_6 = 28$, because each term increases by 5.

n	1	2	3	4	5	6
u_n	3	8	13	18	23	28

+5 +5 +5 +5 +5

Ex 7: What is u_5 for this sequence?

n	1	2	3	4	5
u_n	20	18	16	14	12

Answer: $u_5 = 12$, because each term decreases by 2.

n	1	2	3	4	5
u_n	20	18	16	14	12

−2 −2 −2 −2

Ex 8: What is u_6 for this sequence?

n	1	2	3	4	5	6
u_n	80	70	60	50	40	30

Answer: $u_6 = 30$, because each term decreases by 10.

n	1	2	3	4	5	6
u_n	80	70	60	50	40	30

−10 −10 −10 −10 −10

B DEFINITION USING A RECURSIVE RULE

B.1 CALCULATING THE FIRST TERMS

Ex 9: Write the sequence defined by: the first term is 7, and each term is obtained by adding 4 to the previous term.

$(\boxed{7}, \boxed{11}, \boxed{15}, \boxed{19}, \boxed{23}, \dots)$

Answer:

$$7 \xrightarrow{+4} 11 \xrightarrow{+4} 15 \xrightarrow{+4} 19 \xrightarrow{+4} 23$$

The sequence is: $(7, 11, 15, 19, 23, \dots)$.

Ex 10: Write the sequence defined by: the first term is 1, and each term is obtained by multiplying the previous term by 2.

$(\boxed{1}, \boxed{2}, \boxed{4}, \boxed{8}, \boxed{16}, \dots)$

Answer:

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16$$

The sequence is: $(1, 2, 4, 8, 16, \dots)$.

Ex 11: Write the sequence defined by: the first term is 10, and each term is obtained by subtracting 5 from the previous term.

$(\boxed{10}, \boxed{5}, \boxed{0}, \boxed{-5}, \boxed{-10}, \dots)$

Answer:

$$10 \xrightarrow{-5} 5 \xrightarrow{-5} 0 \xrightarrow{-5} -5 \xrightarrow{-5} -10$$

The sequence is: $(10, 5, 0, -5, -10, \dots)$.

Ex 12: Write the sequence defined by: the first term is 2.5, and each term is obtained by adding 0.5 to the previous term.

$(\boxed{2.5}, \boxed{3}, \boxed{3.5}, \boxed{4}, \boxed{4.5}, \dots)$

Answer:

$$2.5 \xrightarrow{+0.5} 3 \xrightarrow{+0.5} 3.5 \xrightarrow{+0.5} 4 \xrightarrow{+0.5} 4.5$$

The sequence is: $(2.5, 3, 3.5, 4, 4.5, \dots)$.

B.2 CALCULATING THE FIRST TERMS

Ex 13: Calculate the first terms of the sequence defined by:

$$u_0 = 3 \text{ and } u_{n+1} = u_n + 4.$$

- $u_1 = \boxed{7}$
- $u_2 = \boxed{11}$
- $u_3 = \boxed{15}$
- $u_4 = \boxed{19}$
- $u_5 = \boxed{23}$

Answer:

$$3 \xrightarrow{+4} 7 \xrightarrow{+4} 11 \xrightarrow{+4} 15 \xrightarrow{+4} 19 \xrightarrow{+4} 23$$

- $u_1 = u_0 + 4 = 3 + 4 = 7$
- $u_2 = u_1 + 4 = 7 + 4 = 11$
- $u_3 = u_2 + 4 = 11 + 4 = 15$
- $u_4 = u_3 + 4 = 15 + 4 = 19$
- $u_5 = u_4 + 4 = 19 + 4 = 23$

Ex 14: Calculate the first terms of the sequence defined by:

$$u_0 = 3 \text{ and } u_{n+1} = 2u_n.$$

- $u_1 = \boxed{6}$
- $u_2 = \boxed{12}$
- $u_3 = \boxed{24}$
- $u_4 = \boxed{48}$
- $u_5 = \boxed{96}$

Answer:

$$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{\times 2} 48 \xrightarrow{\times 2} 96$$

- $u_1 = 2u_0 = 2 \times 3 = 6$
- $u_2 = 2u_1 = 2 \times 6 = 12$
- $u_3 = 2u_2 = 2 \times 12 = 24$
- $u_4 = 2u_3 = 2 \times 24 = 48$
- $u_5 = 2u_4 = 2 \times 48 = 96$

Ex 15: Calculate the first terms of the sequence defined by:

$$u_0 = 12 \text{ and } u_{n+1} = u_n - 10.$$

- $u_1 = \boxed{2}$
- $u_2 = \boxed{-8}$
- $u_3 = \boxed{-18}$
- $u_4 = \boxed{-28}$
- $u_5 = \boxed{-38}$

Answer:

$$12 \xrightarrow{-10} 2 \xrightarrow{-10} -8 \xrightarrow{-10} -18 \xrightarrow{-10} -28 \xrightarrow{-10} -38$$

- $u_1 = u_0 - 10 = 12 - 10 = 2$
- $u_2 = u_1 - 10 = 2 - 10 = -8$
- $u_3 = u_2 - 10 = -8 - 10 = -18$
- $u_4 = u_3 - 10 = -18 - 10 = -28$
- $u_5 = u_4 - 10 = -28 - 10 = -38$

Ex 16: Calculate the first terms of the sequence defined by:

$$u_0 = 64 \text{ and } u_{n+1} = \frac{u_n}{2}.$$

- $u_1 = \boxed{32}$
- $u_2 = \boxed{16}$
- $u_3 = \boxed{8}$
- $u_4 = \boxed{4}$
- $u_5 = \boxed{2}$

Answer:

$$64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2$$

- $u_1 = \frac{u_0}{2} = \frac{64}{2} = 32$
- $u_2 = \frac{u_1}{2} = \frac{32}{2} = 16$
- $u_3 = \frac{u_2}{2} = \frac{16}{2} = 8$
- $u_4 = \frac{u_3}{2} = \frac{8}{2} = 4$
- $u_5 = \frac{u_4}{2} = \frac{4}{2} = 2$

B.3 IDENTIFYING THE RECURSIVE RULE

Ex 17: Given the sequence: $(3, 5, 7, 9, 11, 13, \dots)$

- The first term is $\boxed{3}$.
- The rule is $\boxed{\textbf{Add}}$ $\boxed{2}$

Answer:

- The first term is 3.
- The rule is add 2:

$$3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9 \xrightarrow{+2} 11 \xrightarrow{+2} 13$$

Ex 18: Given the sequence: $(60, 55, 50, 45, 40, 35, \dots)$

- The first term is $\boxed{60}$.
- The rule is $\boxed{\textbf{Subtract}}$ $\boxed{5}$

Answer:

- The first term is 60.
- The rule is subtract 5:

$$60 \xrightarrow{-5} 55 \xrightarrow{-5} 50 \xrightarrow{-5} 45 \xrightarrow{-5} 40 \xrightarrow{-5} 35$$

Ex 19: Given the sequence: $(64, 32, 16, 8, 4, 2, \dots)$

- The first term is $\boxed{64}$.
- The rule is $\boxed{\textbf{Divide}}$ $\boxed{2}$

Answer:

- The first term is 64.
- The rule is divide by 2:

$$64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2$$

Ex 20: Given the sequence: $(1, 10, 100, 1\,000, 10\,000, \dots)$

- The first term is $\boxed{1}$.
- The rule is $\boxed{\textbf{Multiply}}$ $\boxed{10}$

Answer:

- The first term is 1.
- The rule is multiply by 10:

$$1 \xrightarrow{\times 10} 10 \xrightarrow{\times 10} 100 \xrightarrow{\times 10} 1\,000 \xrightarrow{\times 10} 10\,000$$

B.4 IDENTIFYING THE RECURSIVE RULE

Ex 21: Given the sequence: $(3, 5, 7, 9, 11, 13, \dots)$, what is its recursive rule?

- $u_0 = \boxed{3}$.
- $u_{n+1} = \boxed{u_n + 2}$

Answer:

- The first term is $u_0 = 3$.
- The rule is add 2:

$$(\ 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \xrightarrow{+2} 9\ , \dots ,\ u_n \xrightarrow{+2} u_{n+1}\ , \dots)$$

$$u_{n+1} = u_n + 2$$

Ex 22: Given the sequence: $(100, 90, 80, 70, 60, \dots)$, what is its recursive rule?

- $u_0 = \boxed{100}$.
- $u_{n+1} = \boxed{u_n - 10}$

Answer:

- The first term is $u_0 = 100$.
- The rule is subtract 10:

$$(\ 100 \xrightarrow{-10} 90 \xrightarrow{-10} 80 \xrightarrow{-10} 70\ , \dots ,\ u_n \xrightarrow{-10} u_{n+1}\ , \dots)$$

$$u_{n+1} = u_n - 10$$

Ex 23: Given the sequence: $(2, 6, 18, 54, 162, \dots)$, what is its recursive rule?

- $u_0 = \boxed{2}$.
- $u_{n+1} = \boxed{3 \times u_n}$

Answer:

- The first term is $u_0 = 2$.
- The rule is multiply by 3:

$$(\ 2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54\ , \dots ,\ u_n \xrightarrow{\times 3} u_{n+1}\ , \dots)$$

$$u_{n+1} = 3 \times u_n$$

Ex 24: Given the sequence: $(8, 4, 2, 1, 0.5, 0.25, \dots)$, what is its recursive rule?

- $u_0 = \boxed{8}$.
- $u_{n+1} = \boxed{u_n \div 2}$

Answer:

- The first term is $u_0 = 8$.
- The rule is divide by 2:

$$(\ 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2 \xrightarrow{\div 2} 1 \xrightarrow{\div 2} 0.5\ , \dots ,\ u_n \xrightarrow{\div 2} u_{n+1}\ , \dots)$$

$$u_{n+1} = \frac{u_n}{2}$$

B.5 MODELING REAL SITUATIONS

Ex 25: A scientist observes a culture of bacteria. At the start, there are $u_0 = 5$ bacteria in a petri dish. Each day, the number of bacteria doubles.
Let u_n be the number of bacteria at the day n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{10}$ bacteria
- $u_2 = \boxed{20}$ bacteria
- $u_3 = \boxed{40}$ bacteria

What is its recursive rule?

$$u_{n+1} = \boxed{2 \times u_n}$$

Answer: The number of bacteria doubles each day:

- $u_1 = 2 \times u_0 = 2 \times 5 = 10$
- $u_2 = 2 \times u_1 = 2 \times 10 = 20$
- $u_3 = 2 \times u_2 = 2 \times 20 = 40$

The rule is multiply by 2:

$$u_{n+1} = 2 \times u_n$$

Ex 26: Each day, I walk more and more steps. On day 0, I walk $u_0 = 1000$ steps. Each day, I walk 500 steps more than the day before.
Let u_n be the number of steps I have walked at the end of day n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{1500}$ steps
- $u_2 = \boxed{2000}$ steps
- $u_3 = \boxed{2500}$ steps

What is its recursive rule?

$$u_{n+1} = \boxed{u_n + 500}$$

Answer: The number of steps increases by 500 each day:

- $u_1 = u_0 + 500 = 1000 + 500 = 1500$
- $u_2 = u_1 + 500 = 1500 + 500 = 2000$
- $u_3 = u_2 + 500 = 2000 + 500 = 2500$

The recursive rule is:

$$u_{n+1} = u_n + 500$$

Ex 27: Suppose I deposit \$100 in a savings account. Each year, my amount is multiplied by 1.1 (that is, it increases by 10% each year).
Let u_n be the amount of money in the account after n years. What are the first three terms of the sequence (u_n)?

- $u_0 = \boxed{100}$ dollars
- $u_1 = \boxed{110}$ dollars
- $u_2 = \boxed{121}$ dollars

What is its recursive rule?

$$u_{n+1} = \boxed{1.1 \times u_n}$$

Answer: The amount increases by 10% each year, so it is multiplied by 1.1:

- $u_0 = 100$
- $u_1 = 1.1 \times u_0 = 1.1 \times 100 = 110$
- $u_2 = 1.1 \times u_1 = 1.1 \times 110 = 121$

The recursive rule is:

$$u_{n+1} = 1.1 \times u_n$$

Ex 28: At the start, I have $u_0 = 20$ dollars. Each week, my parents give me \$10 more.
Let u_n be the amount of money I have at the beginning of week n. What are the first three terms of the sequence (u_n)?

- $u_1 = \boxed{30}$ dollars
- $u_2 = \boxed{40}$ dollars
- $u_3 = \boxed{50}$ dollars

What is its recursive rule?

$$u_{n+1} = \boxed{u_n + 10}$$

Answer: The sequence increases by \$10 each week:

- $u_1 = u_0 + 10 = 20 + 10 = 30$
- $u_2 = u_1 + 10 = 30 + 10 = 40$
- $u_3 = u_2 + 10 = 40 + 10 = 50$

The recursive rule is:

$$u_{n+1} = u_n + 10$$

C DEFINITION USING AN EXPLICIT RULE

C.1 CALCULATING TERMS FROM AN EXPLICIT FORMULA

Ex 29: Consider the sequence defined by the explicit formula: $u_n = 3n + 2$.
Write the first four terms of this sequence.

- $u_0 = \boxed{2}$
- $u_1 = \boxed{5}$
- $u_2 = \boxed{8}$
- $u_3 = \boxed{11}$

Answer:

- For $n=0$:

$$\begin{aligned} u_0 &= 3\times 0+2 \\ &= 0+2 \\ &= 2 \end{aligned}$$

- For $n=1$:

$$\begin{aligned} u_1 &= 3\times 1+2 \\ &= 3+2 \\ &= 5 \end{aligned}$$

- For $n=2$:

$$\begin{aligned} u_2 &= 3\times 2+2 \\ &= 6+2 \\ &= 8 \end{aligned}$$

- For $n=3$:

$$\begin{aligned} u_3 &= 3\times 3+2 \\ &= 9+2 \\ &= 11 \end{aligned}$$

So the first four terms are: $2,\ 5,\ 8,\ 11$.

Ex 30: Consider the sequence defined by the explicit formula: $u_n = -10n+100$.
Write the first four terms of this sequence.

- $u_0 = \boxed{100}$
- $u_1 = \boxed{90}$
- $u_2 = \boxed{80}$
- $u_3 = \boxed{70}$

Answer:

- For $n=0$:

$$\begin{aligned} u_0 &= -10\times 0+100 \\ &= 0+100 \\ &= 100 \end{aligned}$$

- For $n=1$:

$$\begin{aligned} u_1 &= -10\times 1+100 \\ &= -10+100 \\ &= 90 \end{aligned}$$

- For $n=2$:

$$\begin{aligned} u_2 &= -10\times 2+100 \\ &= -20+100 \\ &= 80 \end{aligned}$$

- For $n=3$:

$$\begin{aligned} u_3 &= -10\times 3+100 \\ &= -30+100 \\ &= 70 \end{aligned}$$

So the first four terms are: $100,\ 90,\ 80,\ 70$.

Ex 31: Consider the sequence defined by the explicit formula: $u_n = n^2+2$.
Write the first four terms of this sequence.

- $u_0 = \boxed{2}$
- $u_1 = \boxed{3}$
- $u_2 = \boxed{6}$
- $u_3 = \boxed{11}$

Answer:

- For $n=0$:

$$\begin{aligned} u_0 &= 0^2+2 \\ &= 0+2 \\ &= 2 \end{aligned}$$

- For $n=1$:

$$\begin{aligned} u_1 &= 1^2+2 \\ &= 1+2 \\ &= 3 \end{aligned}$$

- For $n=2$:

$$\begin{aligned} u_2 &= 2^2+2 \\ &= 4+2 \\ &= 6 \end{aligned}$$

- For $n=3$:

$$\begin{aligned} u_3 &= 3^2+2 \\ &= 9+2 \\ &= 11 \end{aligned}$$

So the first four terms are: $2,\ 3,\ 6,\ 11$.

Ex 32: Consider the sequence defined by the explicit formula: $u_n = (n+1)n$.
Write the first four terms of this sequence.

- $u_0 = \boxed{0}$
- $u_1 = \boxed{2}$
- $u_2 = \boxed{6}$
- $u_3 = \boxed{12}$

Answer:

- For $n=0$:

$$\begin{aligned} u_0 &= (0+1)\times 0 \\ &= 1\times 0 \\ &= 0 \end{aligned}$$

- For $n=1$:

$$\begin{aligned} u_1 &= (1+1)\times 1 \\ &= 2\times 1 \\ &= 2 \end{aligned}$$

- For $n=2$:

$$\begin{aligned} u_2 &= (2+1)\times 2 \\ &= 3\times 2 \\ &= 6 \end{aligned}$$

- For $n=3$:

$$\begin{aligned} u_3 &= (3+1)\times 3 \\ &= 4\times 3 \\ &= 12 \end{aligned}$$

So the first four terms are: $0,\ 2,\ 6,\ 12$.

C.2 MODELING REAL SITUATIONS WITH SEQUENCES

Ex 33: You start with \$30 and each week your parent gives you \$10.
The amount of money you have after n weeks is given by the formula:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr weeks} \times \text{Amount received each week} \\ &= 30 + n \times 10 \\ &= 30 + 10n \end{aligned}$$

where u_n is the amount after n weeks. How much money will you have after 20 weeks?

$\boxed{230}$ dollars

Answer:

- After 20 weeks:

$$\begin{aligned} u_{20} &= 30 + 10 \times 20 \\ &= 30 + 200 \\ &= 230 \end{aligned}$$

So, after 20 weeks you will have \$230.

Ex 34: You deposit \$1 500 in a savings account that pays simple interest at a rate of 4% per year.
The amount of money in your account after n years is given by the formula:

$$\begin{aligned} u_n &= \text{Initial Amount} + \text{Nbr years} \times \text{Percentage of the initial amount} \\ &= 1\,500 + n \times 0.04 \times 1\,500 \\ &= 1\,500 + 60n \end{aligned}$$

where u_n is the amount after n years. What is your amount at year 20?

$\boxed{2700}$ dollars

Answer:

- At year 20:

$$\begin{aligned} u_{20} &= 1\,500 + 60 \times 20 \\ &= 1\,500 + 1\,200 \\ &= 2\,700 \end{aligned}$$

So, your amount at year 20 is \$2 700.

Ex 35: You start a stamp collection with 12 stamps. Each month, you add 4 new stamps to your collection.
The number of stamps you have after n months is given by the formula:

$$\begin{aligned} u_n &= \text{Initial number of stamps} + \text{Nbr months} \times \text{Stamps added per month} \\ &= 12 + n \times 4 \\ &= 12 + 4n \end{aligned}$$

where u_n is the number of stamps after n months. How many stamps will you have after 15 months?

$\boxed{72}$ stamps

Answer:

- After 15 months:

$$\begin{aligned} u_{15} &= 12 + 4 \times 15 \\ &= 12 + 60 \\ &= 72 \end{aligned}$$

So, after 15 months you will have 72 stamps.

Ex 36: A school plants 5 trees in its garden to start. Every year, they plant 3 new trees.
The total number of trees after n years is given by the formula:

$$\begin{aligned} u_n &= \text{Initial number of trees} + \text{Nbr years} \times \text{Trees planted per year} \\ &= 5 + n \times 3 \\ &= 5 + 3n \end{aligned}$$

where u_n is the number of trees after n years. How many trees will there be after 12 years?

$\boxed{41}$ trees

Answer:

- After 12 years:

$$\begin{aligned} u_{12} &= 5 + 3 \times 12 \\ &= 5 + 36 \\ &= 41 \end{aligned}$$

So, after 12 years there will be 41 trees.