SOLVING QUADRATIC EQUATIONS

The purpose of this section is to learn how to solve quadratic equations.

A DEFINITION

Definition Quadratic Equation

Given numbers a, b, and c with $a \neq 0$, a **quadratic equation** is any equation of the form

$$ax^2 + bx + c = 0.$$

Ex: Is $3x^2 + 5x + 4$ a quadratic polynomial? If yes, identify the coefficients a, b, and c.

Answer: Yes: $a = 3$, $b = 5$, $c = 4$.

Definition Root

A **root** of the equation $ax^2 + bx + c = 0$ is any number that, when substituted for x, makes the equation true.

Ex: Are 1 and 3 roots of the equation $x^2 - 3x + 2 = 0$?

Answer: To check if 1 and 3 are roots, substitute each into the equation:

- For $x = 1$, $1^2 - 3 \cdot 1 + 2 = 1 - 3 + 2 = 0$.
  So 1 is a root.
- For $x = 3$, $3^2 - 3 \cdot 3 + 2 = 9 - 9 + 2 = 2 \neq 0$.
  So 3 is not a root

A quadratic equation may have no real solution. For example, $x^2 = -1$ has no real solution because the square of a real number cannot be negative.

B SOLVING BY FACTORIZATION

To solve a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$, we can leverage our understanding of solving linear equations. The key idea is to transform the quadratic equation into a product of linear factors using **factorization**. This allows us to convert the problem into solving simpler linear equations, which we already know how to handle.

Proposition Null Factor Law

If $ab = 0$, then $a = 0$ or $b = 0$.

Ex: Solve $(x - 1)(x + 2) = 0$.

Answer:

$$\begin{aligned} (x-1)(x+2) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad x + 2 = 0 \quad \text{(null factor law)} \\ x = 1 \quad &\text{or} \quad x = -2 \quad \text{(solve each equation)} \end{aligned}$$

Method Solving by Factorization

To solve a quadratic equation:

1. **Factorize**,
2. **Apply the null factor law**,
3. **Solve the resulting linear equations**.

This turns the problem of solving a quadratic equation into finding a **factorization**.

C FACTORIZATION TECHNIQUES FOR SPECIAL FORMS OF EQUATIONS

Proposition Common Factor Law for equations of the form $x^2 + ax$

$$x^2 + ax = x(x + a)$$

Ex: Find the roots of $x^2 - 2x = 0$.

Answer:

$$\begin{aligned} x^2 - 2x &= 0 \\ x(x-2) &= 0 && \text{(factorizing)} \\ x = 0 \quad \text{or} \quad x - 2 &= 0 && \text{(null factor law)} \\ x = 0 \quad \text{or} \quad x &= 2 && \text{(solving linear equations)} \end{aligned}$$

Proposition **Perfect Square for equations of the form $x^2 + 2ax + a^2$**

$$x^2 + 2ax + a^2 = (x+a)^2$$

Ex: Solve $x^2 + 2x + 1 = 0$.

Answer:

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x+1)^2 &= 0 && \text{(perfect square)} \\ x + 1 &= 0 && \text{(null factor law)} \\ x &= -1 && \text{(solving linear equation)} \end{aligned}$$

So, -1 is a double root.

Proposition **Difference of Squares for equations of the form $x^2 - a^2$**

$$x^2 - a^2 = (x-a)(x+a)$$

Ex: Solve $x^2 - 9 = 0$.

Answer:

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 && \text{(difference of squares)} \\ x - 3 = 0 \quad \text{or} \quad x + 3 &= 0 && \text{(null factor law)} \\ x = 3 \quad \text{or} \quad x &= -3 && \text{(solving linear equations)} \end{aligned}$$

D FACTORIZATION BY COMPLETING THE SQUARE

Some quadratics like $x^2 + 2x - 3$ cannot be factored easily. In this case, we use **completing the square**.

Proposition **Completing the Square**

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2.$$

Proof

$$\begin{aligned} \left(x + \frac{b}{2}\right)^2 &= x^2 + bx + \left(\frac{b}{2}\right)^2 && \text{(perfect square)} \\ \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c && \text{(adding } -\left(\frac{b}{2}\right)^2 + c \text{ to both sides)} \\ \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c &= x^2 + bx + c && \text{(simplifying).} \end{aligned}$$

Ex: Complete the square for $x^2 + 10x + 24 = 0$.

Answer: We know that $(x+5)^2 = x^2 + 10x + 25$. So

$$\begin{aligned} x^2 + 10x + 24 &= 0 \\ x^2 + 10x + 25 - 1 &= 0 && \text{(rewrite 24 as } 25 - 1\text{)} \\ (x+5)^2 - 1 &= 0 && \text{(complete the square).} \end{aligned}$$

Method **General method to solve quadratic equations**

- Step 1: **Complete the square**
- Step 2: **Use the difference of squares**

- Step 3: **Apply the null factor law**
- Step 4: **Solve the linear equations**

Ex: Solve $x^2+10x+24=0$.

Answer: We know that $(x+5)^2=x^2+10x+25$. So

$$\begin{aligned}
x^2+10x+24&=0\\
x^2+10x+25-1&=0 &&\text{(rewrite 24 as } 25-1)\\
(x+5)^2-1&=0 &&\text{(complete the square)}\\
(x+5)^2-1^2&=0 &&\text{(difference of squares)}\\
(x+5-1)(x+5+1)&=0 &&\text{(factorize)}\\
(x+4)(x+6)&=0 &&\text{(simplify)}\\
x+4=0 \text{ or } x+6&=0 &&\text{(null factor law)}\\
x=-4 \text{ or } x&=-6 &&\text{(solve).}
\end{aligned}$$

E QUADRATIC FORMULA

Definition **Discriminant**

Given a quadratic equation $ax^2+bx+c=0$, the discriminant, denoted Δ, is defined as

$$\Delta=b^2-4ac.$$

Proposition **Quadratic formula**

For any quadratic equation $ax^2+bx+c=0$:

- If $\Delta>0$, there are two real roots:
$$x=\frac{-b-\sqrt{\Delta}}{2a} \text{ or } x=\frac{-b+\sqrt{\Delta}}{2a}$$
- If $\Delta=0$, there is one real root:
$$x=\frac{-b}{2a}.$$
- If $\Delta<0$, there are no real roots.

Proof

Suppose $ax^2+bx+c=0$, where $a\neq 0$.

$$\begin{aligned}
ax^2+bx+c&=0\\
\therefore x^2+\frac{b}{a}x+\frac{c}{a}&=0 &&\text{(divide each term by } a\text{, since } a\neq 0)\\
\therefore x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}&=0 &&\text{(complete the square)}\\
\therefore \left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}&=0 &&\text{(simplify)}\\
\therefore \left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}&=0 &&\text{(where } \Delta=b^2-4ac).
\end{aligned}$$

Now, consider the cases based on the discriminant Δ:

- **Case** $\Delta\geq 0$: Since $\frac{\Delta}{4a^2}\geq 0$, a real square root exists.

$$\begin{aligned}
\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{\Delta}{4a^2}}\right)^2&=0\\
\therefore \left(x+\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}\right)\left(x+\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}}\right)&=0 &&\text{(difference of squares).}
\end{aligned}$$

Applying the null factor law:

$$x+\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}=0 \quad \text{or} \quad x+\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}}=0.$$

Solving these linear equations:

$$x=-\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}} \quad \text{or} \quad x=-\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}.$$

Simplifying:

$$x=\frac{-b\pm\sqrt{\Delta}}{2a}.$$

- If $\Delta > 0$, there are two distinct real roots.
- If $\Delta = 0$, there is one real root (double root): $x=-\frac{b}{2a}$.

- **Case** $\Delta < 0$: Then $\frac{\Delta}{4a^2} < 0$, so

$$\left(x+\frac{b}{2a}\right)^2=\frac{\Delta}{4a^2}<0.$$

Since the square of a real number is non-negative, there are no real solutions.

Ex: Consider the quadratic equation $x^2+2x-3=0$.

1. Find the discriminant.
2. Hence, state the nature of the roots of the equation.
3. Solve the equation.

Answer: $x^2+2x-3=0$ has $a=1$, $b=2$, $c=-3$.

1. $\Delta = b^2-4ac$
   $= (2)^2-4(1)(-3)$
   $= 4+12$
   $= 16$
2. As $\Delta > 0$, there are 2 distinct roots.
3. $x=\frac{-b-\sqrt{\Delta}}{2a}$ or $x=\frac{-b+\sqrt{\Delta}}{2a}$
   $x=\frac{-2-\sqrt{16}}{2\cdot 1}$ or $x=\frac{-2+\sqrt{16}}{2\cdot 1}$
   $x=\frac{-2-4}{2}$ or $x=\frac{-2+4}{2}$
   $x=-3$ or $x=1$