A SYSTEMS OF LINEAR EQUATIONS

A.1 VERIFYING SOLUTIONS TO LINEAR EQUATIONS

MCQ 1: Which ordered pair (x, y) is the solution to the system of equations:

$$\begin{cases} 2x - 3y = 1 \\ x + 2y = 4 \end{cases}$$

☐ $(1, 1)$

☐ $(2, 1)$

☐ $(-1, -1)$

☐ $(2, -1)$

MCQ 2: Which ordered triple (x, y, z) is the solution to the system of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ x - 2y + 3z = 6 \end{cases}$$

☐ $(1, 2, 3)$

☐ $(3, 2, 1)$

☐ $(2, 2, 2)$

☐ $(4, 3, -1)$

MCQ 3: Which ordered pair (x, y) is **not** a solution to the equation $x + y = 2$?

☐ $(1, 1)$

☐ $(4, -2)$

☐ $(2, 1)$

☐ $(-1, 3)$

A.2 IDENTIFYING LINEAR EQUATIONS

MCQ 4: Is the equation $2x + xy - 3y = 2$ a linear equation?

☐ Yes

☐ No

MCQ 5: Is the equation $2x + 2 = y$ a linear equation?

☐ Yes

☐ No

MCQ 6: Is the equation $x^2 + 3y = 7$ a linear equation?

☐ Yes

☐ No

MCQ 7: Is the equation $\frac{x+y}{2} = 1$ a linear equation?

☐ Yes

☐ No

MCQ 8: Is the equation $\frac{4}{x} + 2y = 5$ a linear equation?

☐ Yes

☐ No

MCQ 9: Is the equation $x - 2\sqrt{y} = 3$ a linear equation?

☐ Yes

☐ No

A.3 SETTING UP LINEAR SYSTEMS

Ex 10: A father is talking to his son. "The sum of our ages is 50, and the difference between our ages is 28." Let x be the father's age and y be the son's age. Write a system of linear equations that represents this situation.

Ex 11: On a farm, there are chickens and rabbits. The total number of heads is 100 and the total number of legs is 320. Let x be the number of chickens and y be the number of rabbits. Write a system of linear equations that represents this situation.

Ex 12: The perimeter of a rectangular garden is 34 meters. The length of the garden is 5 meters more than its width. Let x be the length and y be the width of the garden. Write a system of linear equations that represents this situation.

Ex 13: A student buys a total of 10 items, consisting of pens and notebooks. Pens cost \$2 each and notebooks cost \$4 each. The total cost of the purchase was \$28. Let x be the number of pens and y be the number of notebooks. Write a system of linear equations that represents this situation.

B REPRESENTATIONS OF SOLUTION SETS

B.1 IDENTIFYING PARAMETRIC SOLUTIONS

MCQ 14: Which of the following is the set of solutions for the equation $2x + 5y = 7$?

- ☐ $x = 1, y = 1$
- ☐ $x = 1 + 5t, y = 1 + 2t$, for all $t \in \mathbb{R}$
- ☐ $x = t, y = \frac{7-2t}{5}$, for all $t \in \mathbb{R}$
- ☐ $x = 6, y = -1 + 2t$, for all $t \in \mathbb{R}$

MCQ 15: Which of the following is the set of solutions for the equation $2x - 2y = -1$?

- ☐ $x = \frac{1}{2}, y = 1$
- ☐ $x = t, y = t - \frac{1}{2}$, for all $t \in \mathbb{R}$
- ☐ $x = t, y = t + \frac{1}{2}$, for all $t \in \mathbb{R}$
- ☐ $x = 1 + t, y = 2 + t$, for all $t \in \mathbb{R}$

MCQ 16: Which of the following is the set of solutions for the equation $3x + 2y = 4$?

- ☐ $x = 2, y = -1$
- ☐ $x = 2t, y = 2 + 3t$, for all $t \in \mathbb{R}$
- ☐ $x = 2 + 3t, y = -1 - 2t$, for all $t \in \mathbb{R}$
- ☐ $x = 2t, y = 2 - 3t$, for all $t \in \mathbb{R}$

B.2 FINDING PARAMETRIC SOLUTIONS

Ex 17: Find the solution set in parametric form for $2x+4y = 1$.

Ex 18: Find the solution set in parametric form for $3x - y = 2$.

Ex 19: Find the solution set in parametric form for $x + 3y = 6$.

B.3 IDENTIFYING PARAMETRIC SOLUTIONS OF SYSTEMS

MCQ 20: Which of the following describes the set of solutions of the plane $x + 2y + z = 4$ in $\mathbb{R}^3$?

- ☐ $x = 1,\ y = 1,\ z = 1$
- ☐ $x = t,\ y = 2 - t,\ z = 2$, for all $t \in \mathbb{R}$
- ☐ $x = 4 - 2s - t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$
- ☐ $x = 4 + 2s + t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$

MCQ 21: Which of the following describes the set of solutions of the plane $x + 2y + 3z = 6$ in $\mathbb{R}^3$?

- ☐ $x = 1,\ y = 1,\ z = 1$
- ☐ $x = 6 - 2s - 3t,\ y = s,\ z = t + 1$, for all $s, t \in \mathbb{R}$
- ☐ $x = 6 - 2s - 3t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$
- ☐ $x = t,\ y = s,\ z = 6 - t - 2s$, for all $s, t \in \mathbb{R}$

MCQ 22: Which of the following describes the set of solutions for the system of equations:

$$\begin{cases} x + y + z = 3 \\ x - y + 2z = 2 \end{cases}$$

- ☐ $x = 1, y = 1, z = 1$
- ☐ $x = \frac{5}{2} + \frac{3}{2}t, \quad y = \frac{1}{2} - \frac{1}{2}t, \quad z = t$, for all $t \in \mathbb{R}$
- ☐ $x = t, y = 1 - t, z = 2$, for all $t \in \mathbb{R}$
- ☐ $x = \frac{5}{2} - \frac{3}{2}t, \quad y = \frac{1}{2} + \frac{1}{2}t, \quad z = t$, for all $t \in \mathbb{R}$

MCQ 23: Which of the following describes the set of solutions for the system of equations:

$$\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 8 \end{cases}$$

☐ $x = 3, y = 0, z = 2$

☐ $x = 13 - 5t, \quad y = 3t - 6, \quad z = t$, for all $t \in \mathbb{R}$

☐ $x = 1, y = 1, z = 2$

☐ $x = 13 + 5t, \quad y = -6 - 3t, \quad z = t$, for all $t \in \mathbb{R}$

C AUGMENTED MATRICES

C.1 WRITING AUGMENTED MATRICES

Ex 24: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} 3x + y = 7 \\ x - 4y = -2 \end{cases}$$

Ex 25: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} 2x - y + z = -1 \\ x + 3z = 4 \\ -x + 2y - z = 0 \end{cases}$$

Ex 26: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} x + 2y - z = 5 \\ -3x + 4z = 0 \end{cases}$$

Ex 27: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} x - y = 3 \\ 2x = 10 \\ -x + 4y = 1 \end{cases}$$

C.2 WRITING SYSTEMS FROM AUGMENTED MATRICES

Ex 28: Write the system of linear equations corresponding to the augmented matrix, using variables x and y.

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 3 & -6 \end{array}\right]$$

Ex 29: Write the system of linear equations corresponding to the augmented matrix, using variables x, y, and z.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Ex 30: Write the system of linear equations corresponding to the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Ex 31: Write the system of linear equations corresponding to the augmented matrix, using variables x, y, and z.

$$\left[\begin{array}{ccc|c} -1 & 2 & 2 & -2 \\ 1 & -1 & 5 & 5 \end{array}\right]$$

D ELEMENTARY ROW OPERATIONS

D.1 PERFORMING ROW OPERATIONS

Ex 32: Write the equivalent system with the first row multiplied by 2.

$$\begin{cases} x - 3y = 1 \\ 2x + y = -1 \end{cases}$$

Ex 33: Write the equivalent system after applying the row operation $R_2 \leftarrow R_2 - 2R_1$.

$$\begin{cases} x - 3y = 1 & (R_1) \\ 2x + y = -1 & (R_2) \end{cases}$$

Ex 34: Write the equivalent system after swapping Row 1 and Row 2 ($R_1 \leftrightarrow R_2$).

$$\begin{cases} 4x + 2y = 10 & (R_1) \\ x - 5y = -13 & (R_2) \end{cases}$$

Ex 35: Write the equivalent system after applying the row operation $R_2 \leftarrow R_2 + 3R_1$.

$$\begin{cases} x + 2y = 4 & (R_1) \\ -3x + 5y = 1 & (R_2) \end{cases}$$

D.2 PERFORMING ROW OPERATIONS (MATRICES)

Ex 36: Given the augmented matrix below, write the equivalent matrix after multiplying the first row by 2 ($R_1 \leftarrow 2R_1$).

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -1 \end{array}\right]$$

Ex 37: Given the augmented matrix below, write the equivalent matrix after applying the row operation $R_2 \leftarrow R_2 - 2R_1$.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -1 \end{array}\right]$$

Ex 38: Given the augmented matrix below, write the equivalent matrix after swapping Row 1 and Row 2 ($R_1 \leftrightarrow R_2$).

$$\left[\begin{array}{cc|c} 4 & 2 & 10 \\ 1 & -5 & -13 \end{array}\right]$$

Ex 39: Given the augmented matrix below, write the equivalent matrix after applying the row operation $R_2 \leftarrow R_2 + 3R_1$.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ -3 & 5 & 1 \end{array}\right]$$

E GAUSSIAN ELIMINATION

E.1 SOLVING 2×2 SYSTEMS OF LINEAR EQUATIONS: LEVEL 1

Ex 40: Use Gaussian elimination to solve:

$$\begin{cases} x - 2y = 8 \\ 3x + y = 3 \end{cases}$$

$x =$ ☐ and $y =$ ☐

Ex 41: Use Gaussian elimination to solve:

$$\begin{cases} x + 3y = 5 \\ 2x + y = 5 \end{cases}$$

$x =$ ☐ and $y =$ ☐

Ex 42: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 3y = 13 \\ x + 4y = 14 \end{cases}$$

$x =$ ☐ and $y =$ ☐

Ex 43: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 3y = 7 \\ x - y = 1 \end{cases}$$

$x =$ ☐ and $y =$ ☐

Ex 44: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 5y = 16 \\ x + 2y = 7 \end{cases}$$

$x =$ ☐ and $y =$ ☐

E.2 SOLVING 3×3 SYSTEMS OF LINEAR EQUATIONS: LEVEL 1

Ex 45: Use Gaussian elimination to solve:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ x + 2y + 3z = 14 \end{cases}$$

$x =$ ☐, $y =$ ☐, $z =$ ☐

Ex 46: Use Gaussian elimination to solve:

$$\begin{cases} x + 2y + z = 7 \\ 2x - y + z = 6 \\ 3x + y - 2z = 1 \end{cases}$$

$x =$ ☐, $y =$ ☐, $z =$ ☐

Ex 47: Use Gaussian elimination to solve:

$$\begin{cases} x + y + 2z = 9 \\ 2x + 3y + z = 11 \\ -x + y + z = 4 \end{cases}$$

$x =$ ☐, $y =$ ☐, $z =$ ☐

Ex 48: Use Gaussian elimination to solve:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 3 \\ x + 2y + 3z = 13 \end{cases}$$

$x =$ ☐, $y =$ ☐, $z =$ ☐

F ANALYZING SOLUTIONS FROM ROW-ECHELON FORM

F.1 ANALYZING 2X2 SYSTEMS WITH PARAMETERS

Ex 49: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} x + ky = 3 \\ 2x + 4y = 1 \end{cases}$$

Determine the values of k for which the system has:

1. a unique solution.
2. no solution.
3. infinitely many solutions.

For the case where a unique solution exists, find the expressions for x and y in terms of k.

Ex 50: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} 2x - y = 4 \\ -6x + 3y = k \end{cases}$$

Determine the values of k for which the system has:

1. a unique solution.
2. no solution.
3. infinitely many solutions.

For the case where infinitely many solutions exist, find the general solution in parametric form.

F.2 ANALYZING 3X3 SYSTEMS WITH PARAMETERS

Ex 51: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} x + y + z = 3 \\ x + 2y - z = 4 \\ 2x + 3y + kz = 8 \end{cases}$$

1. Find the value of k for which the system does not have a unique solution.
2. For this value of k, determine if the system has no solution or infinitely many solutions.
3. Describe the geometric interpretation of the system for this value of k.
4. Given that $k = 1$, find the unique solution to the system.

Ex 52: Consider the following system of linear equations, where $a \in \mathbb{R}$:

$$\begin{cases} x + 2y - z = 1 \\ 2x + 5y + z = 5 \\ x + 3y + 2z = a \end{cases}$$

1. Find the value of a for which the system has infinitely many solutions.
2. For this value of a, find the general solution to the system of equations.
3. Describe the geometric interpretation of the system for this value of a.

Ex 53: Consider the following system of linear equations, where $k, m \in \mathbb{R}$:

$$\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 4 \\ 2x + 3y + kz = m \end{cases}$$

1. Find the value of k for which the system does not have a unique solution.
2. For this value of k, find the value of m for which the system has infinitely many solutions.
3. For the values of k and m found in parts (a) and (b), find the general solution to the system.