SYSTEMS OF LINEAR EQUATIONS

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A.1 VERIFYING SOLUTIONS TO LINEAR EQUATIONS

MCQ 1: Which ordered pair (x, y) is the solution to the system of equations:

$$\begin{cases} 2x - 3y = 1 \\ x + 2y = 4 \end{cases}$$

☐ $(1, 1)$

☒ $(2, 1)$

☐ $(-1, -1)$

☐ $(2, -1)$

Answer: We check each option by substituting the values of x and y into both equations.

- $(1, 1)$**:**
  - Eq 1: $2(1) - 3(1) = 2 - 3 = -1 \neq 1$. (Incorrect)
- $(2, 1)$**:**
  - Eq 1: $2(2) - 3(1) = 4 - 3 = 1$. (Correct)
  - Eq 2: $(2) + 2(1) = 2 + 2 = 4$. (Correct)

  Since both equations are satisfied, this is the solution.
- $(-1, -1)$**:**
  - Eq 1: $2(-1) - 3(-1) = -2 + 3 = 1$. (Correct)
  - Eq 2: $(-1) + 2(-1) = -1 - 2 = -3 \neq 4$. (Incorrect)
- $(2, -1)$**:**
  - Eq 1: $2(2) - 3(-1) = 4 + 3 = 7 \neq 1$. (Incorrect)

MCQ 2: Which ordered triple (x, y, z) is the solution to the system of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ x - 2y + 3z = 6 \end{cases}$$

☒ $(1, 2, 3)$

☐ $(3, 2, 1)$

☐ $(2, 2, 2)$

☐ $(4, 3, -1)$

Answer: We check each option by substituting the values of x, y, and z into all three equations.

- **A:** $(1, 2, 3)$**:**
  - Eq 1: $1 + 2 + 3 = 6$. (Correct)
  - Eq 2: $2(1) + 2 - 3 = 2 + 2 - 3 = 1$. (Correct)
  - Eq 3: $1 - 2(2) + 3(3) = 1 - 4 + 9 = 6$. (Correct)

  Since all three equations are satisfied, this is the solution.
- **B:** $(3, 2, 1)$**:**
  - Eq 1: $3 + 2 + 1 = 6$. (Correct)
  - Eq 2: $2(3) + 2 - 1 = 6 + 2 - 1 = 7 \neq 1$. (Incorrect)
- **C:** $(2, 2, 2)$**:**
  - Eq 1: $2 + 2 + 2 = 6$. (Correct)
  - Eq 2: $2(2) + 2 - 2 = 4 \neq 1$. (Incorrect)
- **D:** $(4, 3, -1)$**:**
  - Eq 1: $4 + 3 + (-1) = 6$. (Correct)
  - Eq 2: $2(4) + 3 - (-1) = 8 + 3 + 1 = 12 \neq 1$. (Incorrect)

MCQ 3: Which ordered pair (x, y) is **not** a solution to the equation $x + y = 2$?

☐ $(1, 1)$

☐ $(4, -2)$

☒ $(2, 1)$

☐ $(-1, 3)$

Answer: We check each option by substituting the values of x and y into the equation.

- $(1, 1)$**:**
  - $1 + 1 = 2$. (This is a solution)
- $(4, -2)$**:**
  - $4 + (-2) = 2$. (This is a solution)
- $(2, 1)$**:**
  - $2 + 1 = 3 \neq 2$. (This is **not** a solution)
- $(-1, 3)$**:**
  - $-1 + 3 = 2$. (This is a solution)

The only ordered pair that is not a solution is $(2, 1)$.

A.2 IDENTIFYING LINEAR EQUATIONS

MCQ 4: Is the equation $2x + xy - 3y = 2$ a linear equation?

☐ Yes

☒ No

Answer: A linear equation is one in which each variable appears in a separate term and is raised to the first power. The term xy involves a product of two variables, which makes the equation non-linear.

MCQ 5: Is the equation $2x + 2 = y$ a linear equation?

☒ Yes

☐ No

Answer: The equation is linear. It can be rearranged into the standard form $ax + by = c$.
Starting from $2x + 2 = y$, subtract y from both sides and then subtract 2 from both sides to obtain $2x - y = -2$. Each variable appears in a separateterm and is raised to the first power.

MCQ 6: Is the equation $x^2 + 3y = 7$ a linear equation?

☐ Yes

☒ No

Answer: The equation is not linear. In a linear equation, all variables must be raised to the first power. The term x^2 violates this condition.

MCQ 7: Is the equation $\frac{x+y}{2} = 1$ a linear equation?

☒ Yes

☐ No

Answer: The equation is linear. By multiplying both sides by 2, we can rewrite it as $x + y = 2$. This is in the standard form $ax + by = c$. The variables are not multiplied together, in a denominator, or raised to a power other than 1.

MCQ 8: Is the equation $\frac{4}{x} + 2y = 5$ a linear equation?

☐ Yes

☒ No

Answer: The equation is not linear. The term $\frac{4}{x}$ can be written as $4x^{-1}$. In a linear equation, all variables must be raised to the first power, not a negative power.

MCQ 9: Is the equation $x - 2\sqrt{y} = 3$ a linear equation?

☐ Yes

☒ No

Answer: The equation is not linear. The term $2\sqrt{y}$ can be written as $2y^{1/2}$. In a linear equation, all variables must be raised to the first power, not a fractional power.

A.3 SETTING UP LINEAR SYSTEMS

Ex 10: A father is talking to his son. "The sum of our ages is 50, and the difference between our ages is 28." Let x be the father's age and y be the son's age. Write a system of linear equations that represents this situation.

Answer: The problem translates into the following system of equations:

$$\begin{cases} x + y = 50 \\ x - y = 28 \end{cases}$$

Ex 11: On a farm, there are chickens and rabbits. The total number of heads is 100 and the total number of legs is 320. Let x be the number of chickens and y be the number of rabbits. Write a system of linear equations that represents this situation.

Answer: Each animal has one head, so the equation for the total number of heads is:

$$x + y = 100$$

Chickens have 2 legs and rabbits have 4 legs, so the equation for the total number of legs is:

$$2x + 4y = 320$$

The problem translates into the following system of equations:

$$\begin{cases} x + y = 100 \\ 2x + 4y = 320 \end{cases}$$

Ex 12: The perimeter of a rectangular garden is 34 meters. The length of the garden is 5 meters more than its width. Let x be the length and y be the width of the garden. Write a system of linear equations that represents this situation.

Answer: The formula for the perimeter of a rectangle is 2(length+ width). This gives the first equation:

$$2x + 2y = 34$$

The statement "the length is 5 meters more than its width" gives the second equation:

$$x = y + 5 \quad \text{or} \quad x - y = 5$$

The problem translates into the following system of equations:

$$\begin{cases} 2x + 2y = 34 \\ x - y = 5 \end{cases}$$

Ex 13: A student buys a total of 10 items, consisting of pens and notebooks. Pens cost \$2 each and notebooks cost \$4 each. The total cost of the purchase was \$28. Let x be the number of pens and y be the number of notebooks. Write a system of linear equations that represents this situation.

Answer: The total number of items gives the first equation:

$$x + y = 10$$

The total cost of the items gives the second equation:

$$2x + 4y = 28$$

The problem translates into the following system of equations:

$$\begin{cases} x + y = 10 \\ 2x + 4y = 28 \end{cases}$$

B REPRESENTATIONS OF SOLUTION SETS

B.1 IDENTIFYING PARAMETRIC SOLUTIONS

MCQ 14: Which of the following is the set of solutions for the equation $2x + 5y = 7$?

☐ $x = 1, y = 1$

☐ $x = 1 + 5t, y = 1 + 2t$, for all $t \in \mathbb{R}$

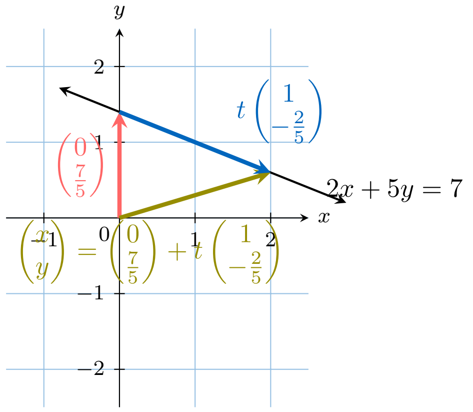
☒ $x = t, y = \frac{7-2t}{5}$, for all $t \in \mathbb{R}$

☐ $x = 6, y = -1 + 2t$, for all $t \in \mathbb{R}$

Answer: We test each proposal by substituting into $2x + 5y$.

- $\mathbf{x = 1,\ y = 1}$: $2(1) + 5(1) = 7$. This is a single solution, *not* the whole set. **Incorrect.**
- $\mathbf{x = 1 + 5t,\ y = 1 + 2t}$: $2(1 + 5t) + 5(1 + 2t) = 7 + 20t \neq 7$ unless $t = 0$. **Incorrect.**
- $\mathbf{x = t,\ y = \frac{7-2t}{5}}$: $2t + 5 \cdot \frac{7-2t}{5} = 7$ for all $t \in \mathbb{R}$. This describes *all* solutions. **Correct.**
- $\mathbf{x = 6,\ y = -1 + 2t}$: $12 + 5(-1 + 2t) = 7 + 10t \neq 7$ unless $t = 0$. **Incorrect.**

Correct choice: $x = t,\ y = \dfrac{7-2t}{5}$.



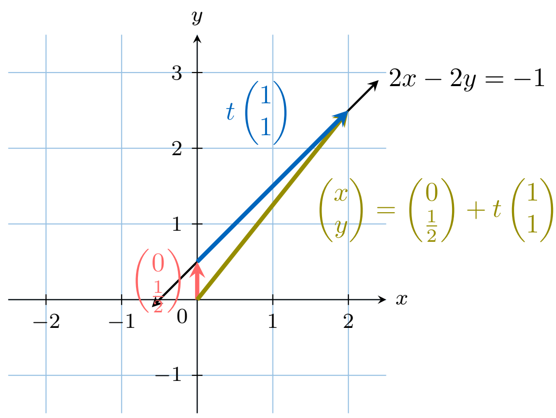
MCQ 15: Which of the following is the set of solutions for the equation $2x - 2y = -1$?

- ☐ $x = \frac{1}{2}, y = 1$
- ☐ $x = t, y = t - \frac{1}{2}$, for all $t \in \mathbb{R}$
- ☒ $x = t, y = t + \frac{1}{2}$, for all $t \in \mathbb{R}$
- ☐ $x = 1 + t, y = 2 + t$, for all $t \in \mathbb{R}$

Answer: We test each proposal by substituting the expressions for x and y into $2x - 2y$.

- $\mathbf{x = \frac{1}{2},\ y = 1}$: $2(\frac{1}{2}) - 2(1) = 1 - 2 = -1$. This is a single solution, *not* the whole set. **Incorrect.**
- $\mathbf{x = t,\ y = t - \frac{1}{2}}$: $2(t) - 2(t - \frac{1}{2}) = 2t - 2t + 1 = 1 \neq -1$. **Incorrect.**
- $\mathbf{x = t,\ y = t + \frac{1}{2}}$: $2(t) - 2(t + \frac{1}{2}) = 2t - 2t - 1 = -1$ for all $t \in \mathbb{R}$. This describes *all* solutions. **Correct.**
- $\mathbf{x = 1 + t,\ y = 2 + t}$: $2(1+t) - 2(2+t) = 2 + 2t - 4 - 2t = -2 \neq -1$. **Incorrect.**

Correct choice: $x = t,\ y = t + \frac{1}{2}$.



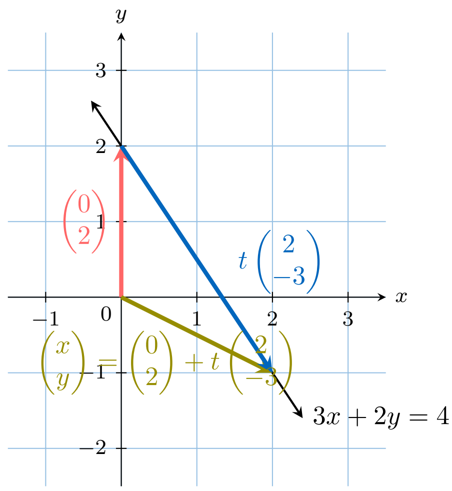
MCQ 16: Which of the following is the set of solutions for the equation $3x + 2y = 4$?

- ☐ $x = 2, y = -1$
- ☐ $x = 2t, y = 2 + 3t$, for all $t \in \mathbb{R}$
- ☐ $x = 2 + 3t, y = -1 - 2t$, for all $t \in \mathbb{R}$
- ☒ $x = 2t, y = 2 - 3t$, for all $t \in \mathbb{R}$

Answer: We test each proposal by substituting the expressions for x and y into $3x + 2y$.

- $\mathbf{x = 2,\ y = -1}$: $3(2) + 2(-1) = 6 - 2 = 4$. This is a single solution, *not* the whole set. **Incorrect.**
- $\mathbf{x = 2t,\ y = 2 + 3t}$: $3(2t)+2(2+3t) = 6t+4+6t = 4+12t \neq 4$ unless $t = 0$. **Incorrect.**
- $\mathbf{x = 2 + 3t,\ y = -1 - 2t}$: $3(2+3t) + 2(-1-2t) = 6 + 9t - 2 - 4t = 4 + 5t \neq 4$ unless $t = 0$. **Incorrect.**
- $\mathbf{x = 2t,\ y = 2 - 3t}$: $3(2t) + 2(2 - 3t) = 6t + 4 - 6t = 4$ for all $t \in \mathbb{R}$. This describes *all* solutions. **Correct.**

Correct choice: $x = 2t,\ y = 2 - 3t$.



B.2 FINDING PARAMETRIC SOLUTIONS

Ex 17: Find the solution set in parametric form for $2x + 4y = 1$.

Answer: Let $x = t$ be a free parameter. Then

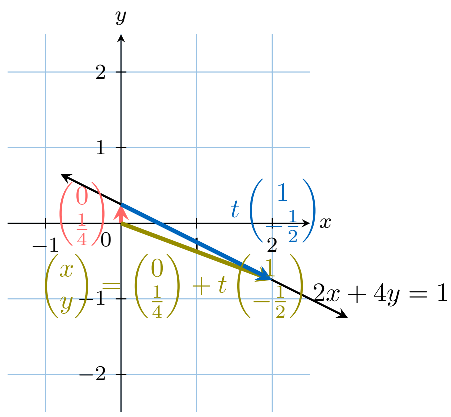
$$2t + 4y = 1 \ \Rightarrow\ 4y = 1 - 2t \ \Rightarrow\ y = \tfrac{1}{4} - \tfrac{1}{2}t.$$

So the parametric solution set is

$$(x, y) = \left(t,\ \tfrac{1}{4} - \tfrac{1}{2}t\right), \quad t \in \mathbb{R}.$$

Equivalently, in vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} + t \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, \quad t \in \mathbb{R}.$$



Ex 18: Find the solution set in parametric form for $3x - y = 2$.

Answer: Let $x = t$ be a free parameter, where $t \in \mathbb{R}$.
Substitute $x = t$ into the equation:

$$3(t) - y = 2$$

Now, solve for y in terms of t:

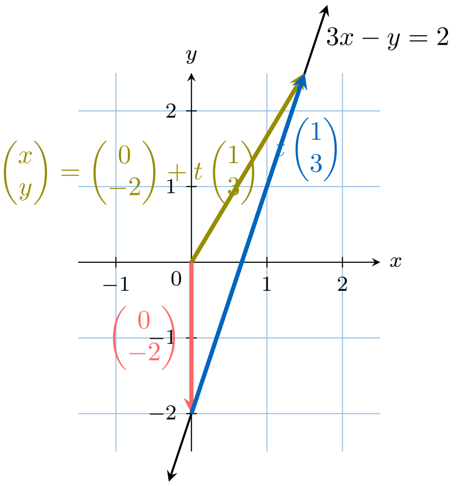
$$y = 3t - 2$$

The solution set in parametric form is:

$$(x, y) = (t, 3t - 2), \quad t \in \mathbb{R}$$

Equivalently, in vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}.$$



Ex 19: Find the solution set in parametric form for $x + 3y = 6$.

Answer: Let $y = t$ be a free parameter, where $t \in \mathbb{R}$.
Substitute $y = t$ into the equation:

$$x + 3(t) = 6$$

Now, solve for x in terms of t:

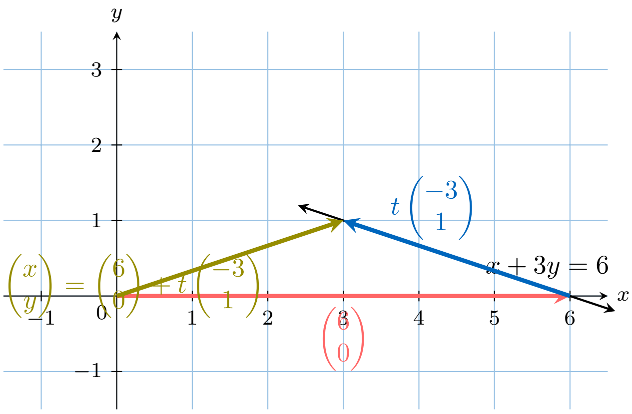
$$x = 6 - 3t$$

The solution set in parametric form is:

$$(x, y) = (6 - 3t, t), \quad t \in \mathbb{R}$$

Equivalently, in vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$



B.3 IDENTIFYING PARAMETRIC SOLUTIONS OF SYSTEMS

MCQ 20: Which of the following describes the set of solutions of the plane $x + 2y + z = 4$ in $\mathbb{R}^3$?

☐ $x = 1,\ y = 1,\ z = 1$

☐ $x = t,\ y = 2 - t,\ z = 2$, for all $t \in \mathbb{R}$

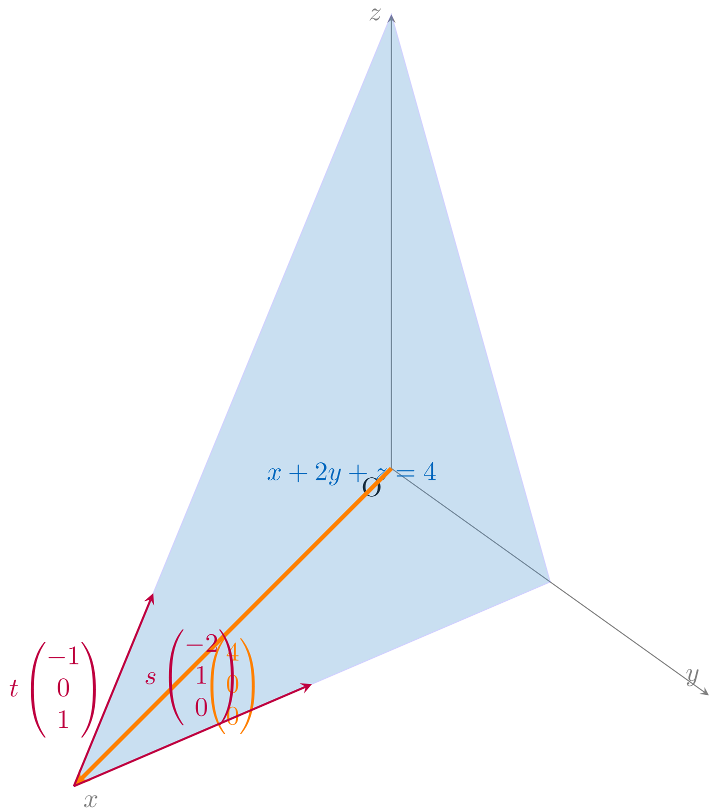
☒ $x = 4 - 2s - t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$

☐ $x = 4 + 2s + t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$

Answer: Substitute each proposal into $x + 2y + z$.

- $(\mathbf{1}, \mathbf{1}, \mathbf{1})$: $1 + 2(1) + 1 = 4$. A single point, not the whole set. **Incorrect**.
- $\mathbf{x = t,\ y = 2 - t,\ z = 2}$: $t + 2(2 - t) + 2 = 6 - t \neq 4$ unless $t = 2$. **Incorrect**.
- $\mathbf{x = 4 - 2s - t,\ y = s,\ z = t}$: $(4 - 2s - t) + 2s + t = 4$ for all s, t. **Correct**.
- $\mathbf{x = 4 + 2s + t,\ y = s,\ z = t}$: $(4 + 2s + t) + 2s + t = 4 + 4s + 2t \neq 4$ in general. **Incorrect**.

Correct choice: $x = 4 - 2s - t,\ y = s,\ z = t$ with $s, t \in \mathbb{R}$.



MCQ 21: Which of the following describes the set of solutions of the plane $x + 2y + 3z = 6$ in $\mathbb{R}^3$?

☐ $x = 1,\ y = 1,\ z = 1$

☐ $x = 6 - 2s - 3t,\ y = s,\ z = t + 1$, for all $s, t \in \mathbb{R}$

☒ $x = 6 - 2s - 3t,\ y = s,\ z = t$, for all $s, t \in \mathbb{R}$

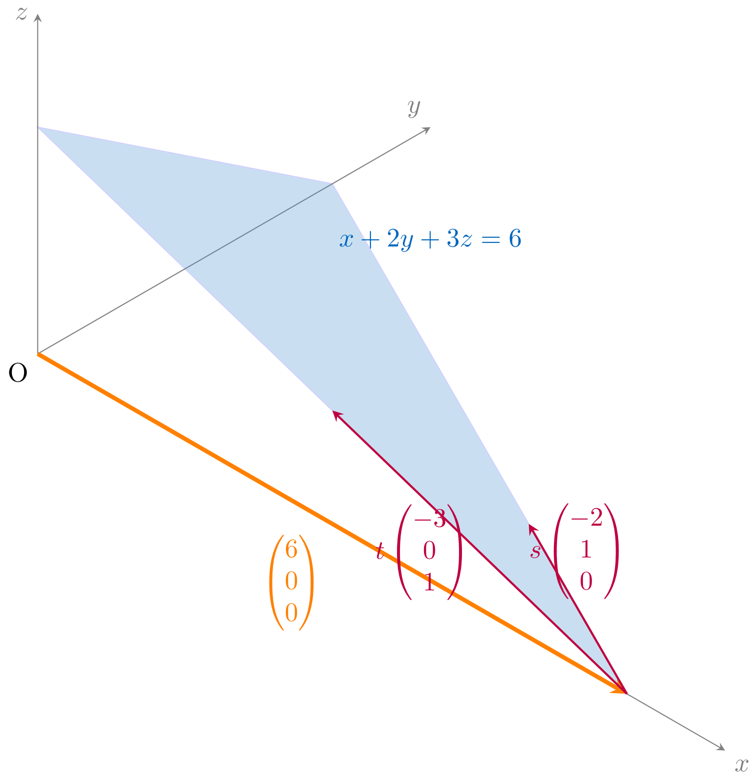
☐ $x = t,\ y = s,\ z = 6 - t - 2s$, for all $s, t \in \mathbb{R}$

Answer: Substitute each proposal into $x + 2y + 3z$.

- $(\mathbf{1},\mathbf{1},\mathbf{1})$: $1+2(1)+3(1)=6$. A single point, not the whole set. **Incorrect**.
- $\mathbf{x=6-2s-3t,\ y=s,\ z=t+1}$: $(6-2s-3t)+2s+3(t+1)=6-2s-3t+2s+3t+3=9\neq 6$. **Incorrect**.
- $\mathbf{x=6-2s-3t,\ y=s,\ z=t}$: $(6-2s-3t)+2s+3t=6$ for all s,t. **Correct**.
- $\mathbf{x=t,\ y=s,\ z=6-t-2s}$: $(t)+2(s)+3(6-t-2s)=t+2s+18-3t-6s=18-2t-4s\neq 6$ in general. **Incorrect**.

Correct choice: $x=6-2s-3t,\ y=s,\ z=t$ with $s,t\in\mathbb{R}$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\underbrace{\begin{pmatrix}6\\0\\0\end{pmatrix}}_{\text{position vector}}+s\underbrace{\begin{pmatrix}-2\\1\\0\end{pmatrix}}_{\text{direction vector 1}}+t\underbrace{\begin{pmatrix}-3\\0\\1\end{pmatrix}}_{\text{direction vector 2}}$$



MCQ 22: Which of the following describes the set of solutions for the system of equations:

$$\begin{cases} x + y + z = 3 \\ x - y + 2z = 2 \end{cases}$$

- ☐ $x=1, y=1, z=1$
- ☐ $x=\frac{5}{2}+\frac{3}{2}t,\quad y=\frac{1}{2}-\frac{1}{2}t,\quad z=t$, for all $t\in\mathbb{R}$
- ☐ $x=t, y=1-t, z=2$, for all $t\in\mathbb{R}$
- ☒ $x=\frac{5}{2}-\frac{3}{2}t,\quad y=\frac{1}{2}+\frac{1}{2}t,\quad z=t$, for all $t\in\mathbb{R}$

Answer: We test each proposal by substituting the expressions for x, y, and z into both equations. A correct parametric solution must satisfy both equations for all values of t.

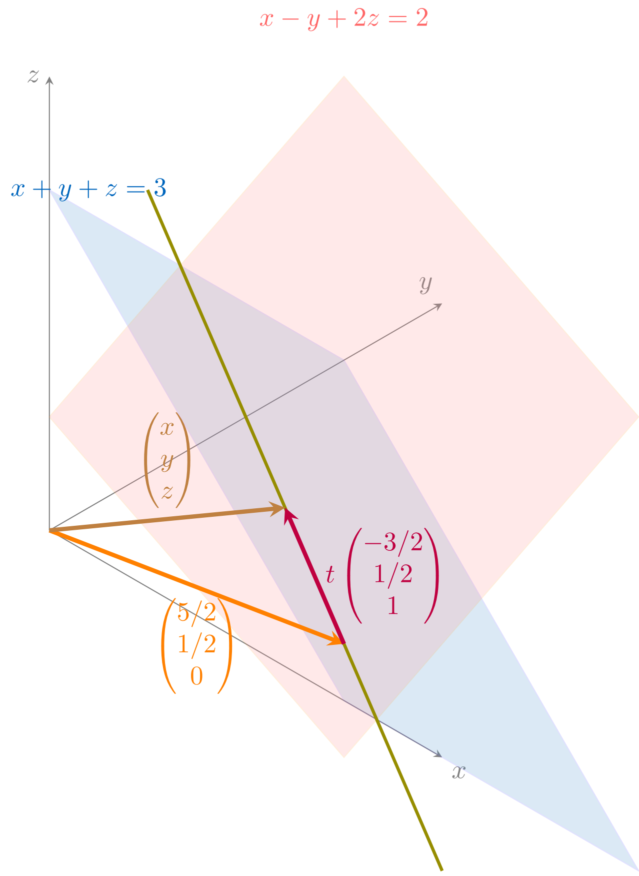
- $(\mathbf{1},\mathbf{1},\mathbf{1})$:
  - Eq 1: $1+1+1=3$. (Correct)
  - Eq 2: $1-1+2(1)=2$. (Correct)

  This is a single point solution, not the whole set of solutions. **Incorrect**.
- $\mathbf{x=\frac{5}{2}+\frac{3}{2}t, y=\frac{1}{2}-\frac{1}{2}t, z=t}$:
  - Eq 1: $(\frac{5}{2}+\frac{3}{2}t)+(\frac{1}{2}-\frac{1}{2}t)+t=3+2t$. This is not equal to 3 for all t. **Incorrect**.
- $\mathbf{x=t, y=1-t, z=2}$:
  - Eq 1: $t+(1-t)+2=3$. (Correct)
  - Eq 2: $t-(1-t)+2(2)=2t-1+4=2t+3$. This is not equal to 2 for all t. **Incorrect**.
- $\mathbf{x=\frac{5}{2}-\frac{3}{2}t, y=\frac{1}{2}+\frac{1}{2}t, z=t}$:
  - Eq 1: $(\frac{5}{2}-\frac{3}{2}t)+(\frac{1}{2}+\frac{1}{2}t)+t=\frac{6}{2}+0t=3$. (Correct for all t)
  - Eq 2: $(\frac{5}{2}-\frac{3}{2}t)-(\frac{1}{2}+\frac{1}{2}t)+2t=\frac{4}{2}+0t=2$. (Correct for all t)

  This describes all solutions. **Correct**.

Correct choice: $x=\frac{5}{2}-\frac{3}{2}t, y=\frac{1}{2}+\frac{1}{2}t, z=t$ with $t\in\mathbb{R}$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5/2-3/2t\\1/2+1/2t\\t\end{pmatrix}=\underbrace{\begin{pmatrix}5/2\\1/2\\0\end{pmatrix}}_{\text{position vector}}+t\underbrace{\begin{pmatrix}-3/2\\1/2\\1\end{pmatrix}}_{\text{direction vector}}$$



MCQ 23: Which of the following describes the set of solutions for the system of equations:

$$\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 8 \end{cases}$$

☐ $x = 3, y = 0, z = 2$

☒ $x = 13 - 5t, \quad y = 3t - 6, \quad z = t$, for all $t \in \mathbb{R}$

☐ $x = 1, y = 1, z = 2$

☐ $x = 13 + 5t, \quad y = -6 - 3t, \quad z = t$, for all $t \in \mathbb{R}$

Answer: We test each proposal by substituting the expressions into both equations. A correct parametric solution must satisfy both equations for all values of t.

- $(\mathbf{3}, \mathbf{0}, \mathbf{2})$:
  - Eq 1: $3 + 2(0) - 2 = 1$. (Correct)
  - Eq 2: $2(3) + 3(0) + 2 = 8$. (Correct)

  This is a single point solution, not the whole set. **Incorrect**.
- $\mathbf{x = 13 - 5t, y = 3t - 6, z = t}$:
  - Eq 1: $(13-5t)+2(3t-6)-t = 13-5t+6t-12-t = 1$. (Correct for all t)
  - Eq 2: $2(13-5t)+3(3t-6)+t = 26-10t+9t-18+t = 8$. (Correct for all t)

  This describes all solutions. **Correct**.
- $(\mathbf{1}, \mathbf{1}, \mathbf{2})$:
  - Eq 1: $1 + 2(1) - 2 = 1$. (Correct)
  - Eq 2: $2(1) + 3(1) + 2 = 7 \neq 8$. (Incorrect).
- $\mathbf{x = 13 + 5t, y = -6 - 3t, z = t}$:
  - Eq 1: $(13+5t)+2(-6-3t)-t = 13+5t-12-6t-t = 1-2t$. This is not equal to 1 for all t. **Incorrect**.

Correct choice: $x = 13 - 5t, y = 3t - 6, z = t$ with $t \in \mathbb{R}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 - 5t \\ -6 + 3t \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 13 \\ -6 \\ 0 \end{pmatrix}}_{\text{position vector}} + t \underbrace{\begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}}_{\text{direction vector}}$$

C AUGMENTED MATRICES

C.1 WRITING AUGMENTED MATRICES

Ex 24: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} 3x + y = 7 \\ x - 4y = -2 \end{cases}$$

Answer: The system has two variables (x, y) and two equations. We write the coefficients of each variable in its own column.
The augmented matrix is:

$$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 1 & -4 & -2 \end{array}\right]$$

Ex 25: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} 2x - y + z = -1 \\ x + 3z = 4 \\ -x + 2y - z = 0 \end{cases}$$

Answer: The system has three variables (x, y, z) and three equations. We write the coefficients of each variable in its own column. If a variable is missing, its coefficient is 0.
The augmented matrix is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 0 & 3 & 4 \\ -1 & 2 & -1 & 0 \end{array}\right]$$

Ex 26: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} x + 2y - z = 5 \\ -3x + 4z = 0 \end{cases}$$

Answer: The system has three variables (x, y, z) and two equations. We write the coefficients of each variable in its own column. The missing y term in the second equation has a coefficient of 0.
The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ -3 & 0 & 4 & 0 \end{array}\right]$$

Ex 27: Write down the augmented matrix for the following system of linear equations:

$$\begin{cases} x - y = 3 \\ 2x = 10 \\ -x + 4y = 1 \end{cases}$$

Answer: The system has two variables (x, y) and three equations. The missing y term in the second equation has a coefficient of 0.
The augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 0 & 10 \\ -1 & 4 & 1 \end{array}\right]$$

C.2 WRITING SYSTEMS FROM AUGMENTED MATRICES

Ex 28: Write the system of linear equations corresponding to the augmented matrix, using variables x and y.

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 3 & -6 \end{array}\right]$$

Answer: The first column represents the coefficients of x, the second column represents the coefficients of y, and the column after the bar represents the constant terms.

- The first row, '[2 -1 | 5]', translates to $2x - 1y = 5$.
- The second row, '[0 3 | -6]', translates to $0x + 3y = -6$.

The corresponding system of equations is:

$$\begin{cases} 2x - y = 5 \\ 3y = -6 \end{cases}$$

Ex 29: Write the system of linear equations corresponding to the augmented matrix, using variables x, y, and z.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Answer: The columns represent the coefficients of x, y, and z respectively.

- The first row translates to $x + 0y - 2z = 9$.
- The second row translates to $0x + y + 5z = -3$.
- The third row translates to $0x + 0y + z = 4$.

The corresponding system of equations is:

$$\begin{cases} x - 2z = 9 \\ y + 5z = -3 \\ z = 4 \end{cases}$$

Ex 30: Write the system of linear equations corresponding to the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Answer: The first column represents the coefficients of x, the second column represents the coefficients of y, and the column after the bar represents the constant terms.

- The first row, '[1 0 | 7]', translates to $1 \cdot x + 0 \cdot y = 7$.
- The second row, '[0 1 | -2]', translates to $0 \cdot x + 1 \cdot y = -2$.

The corresponding system of equations is:

$$\begin{cases} x = 7 \\ y = -2 \end{cases}$$

Ex 31: Write the system of linear equations corresponding to the augmented matrix, using variables x, y, and z.

$$\left[\begin{array}{ccc|c} -1 & 2 & 2 & -2 \\ 1 & -1 & 5 & 5 \end{array}\right]$$

Answer: The first, second, and third columns represent the coefficients of x, y, and z respectively.

- The first row, '[-1 2 2 | -2]', translates to $-x + 2y + 2z = -2$.
- The second row, '[1 -1 5 | 5]', translates to $x - y + 5z = 5$.

The corresponding system of equations is:

$$\begin{cases} -x + 2y + 2z = -2 \\ x - y + 5z = 5 \end{cases}$$

D ELEMENTARY ROW OPERATIONS

D.1 PERFORMING ROW OPERATIONS

Ex 32: Write the equivalent system with the first row multiplied by 2.

$$\begin{cases} x - 3y = 1 \\ 2x + y = -1 \end{cases}$$

Answer: The first row multiplied by 2 is:

$$2 \times (x - 3y) = 2 \times 1$$
$$2x - 6y = 2$$

So the equivalent system is:

$$\begin{cases} 2x - 6y = 2 & (R_1 \leftarrow 2R_1) \\ 2x + y = -1 \end{cases}$$

Ex 33: Write the equivalent system after applying the row operation $R_2 \leftarrow R_2 - 2R_1$.

$$\begin{cases} x - 3y = 1 & (R_1) \\ 2x + y = -1 & (R_2) \end{cases}$$

Answer: We calculate the new second row, $R_2 - 2R_1$:

$$\begin{aligned} (2x + y) - 2(x - 3y) &= -1 - 2(1) \\ 2x + y - 2x + 6y &= -1 - 2 \\ 7y &= -3 \end{aligned}$$

So the equivalent system is:

$$\begin{cases} x - 3y = 1 \\ 7y = -3 & (R_2 \leftarrow R_2 - 2R_1) \end{cases}$$

Ex 34: Write the equivalent system after swapping Row 1 and Row 2 ($R_1 \leftrightarrow R_2$).

$$\begin{cases} 4x + 2y = 10 & (R_1) \\ x - 5y = -13 & (R_2) \end{cases}$$

Answer: Swapping the two rows means the first equation becomes the second, and the second becomes the first.
So the equivalent system is:

$$\begin{cases} x - 5y = -13 \\ 4x + 2y = 10 & (R_1 \leftrightarrow R_2) \end{cases}$$

Ex 35: Write the equivalent system after applying the row operation $R_2 \leftarrow R_2 + 3R_1$.

$$\begin{cases} x + 2y = 4 & (R_1) \\ -3x + 5y = 1 & (R_2) \end{cases}$$

Answer: We calculate the new second row, $R_2 + 3R_1$:

$$\begin{aligned} (-3x + 5y) + 3(x + 2y) &= 1 + 3(4) \\ -3x + 5y + 3x + 6y &= 1 + 12 \\ 11y &= 13 \end{aligned}$$

So the equivalent system is:

$$\begin{cases} x + 2y = 4 \\ 11y = 13 & (R_2 \leftarrow R_2 + 3R_1) \end{cases}$$

D.2 PERFORMING ROW OPERATIONS (MATRICES)

Ex 36: Given the augmented matrix below, write the equivalent matrix after multiplying the first row by 2 ($R_1 \leftarrow 2R_1$).

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -1 \end{array}\right]$$

Answer: Multiplying each element in Row 1 by 2 gives:

$$2 \times 1 = 2, \quad 2 \times (-3) = -6, \quad 2 \times 1 = 2$$

The new augmented matrix is:

$$\left[\begin{array}{cc|c} 2 & -6 & 2 \\ 2 & 1 & -1 \end{array}\right]$$

Ex 37: Given the augmented matrix below, write the equivalent matrix after applying the row operation $R_2 \leftarrow R_2 - 2R_1$.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -1 \end{array}\right]$$

Answer: We calculate the new second row, $R_2 - 2R_1$:

$$\begin{array}{r|l} R_2: & [2 \quad 1 \quad | \quad -1] \\ -2R_1: & [-2 \quad 6 \quad | \quad -2] \\ \hline \text{New } R_2: & [0 \quad 7 \quad | \quad -3] \end{array}$$

The new augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -3 \end{array}\right]$$

Ex 38: Given the augmented matrix below, write the equivalent matrix after swapping Row 1 and Row 2 ($R_1 \leftrightarrow R_2$).

$$\left[\begin{array}{cc|c} 4 & 2 & 10 \\ 1 & -5 & -13 \end{array}\right]$$

Answer: Swapping the two rows means the entries of the first row become the second row, and vice versa.
The new augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & -5 & -13 \\ 4 & 2 & 10 \end{array}\right]$$

Ex 39: Given the augmented matrix below, write the equivalent matrix after applying the row operation $R_2 \leftarrow R_2 + 3R_1$.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ -3 & 5 & 1 \end{array}\right]$$

Answer: We calculate the new second row, $R_2 + 3R_1$:

$$\begin{array}{r|l} R_2: & [-3 \quad 5 \quad | \quad 1] \\ +3R_1: & [3 \quad 6 \quad | \quad 12] \\ \hline \text{New } R_2: & [0 \quad 11 \quad | \quad 13] \end{array}$$

The new augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 11 & 13 \end{array}\right]$$

E GAUSSIAN ELIMINATION

E.1 SOLVING 2×2 SYSTEMS OF LINEAR EQUATIONS: LEVEL 1

Ex 40: Use Gaussian elimination to solve:

$$\begin{cases} x - 2y = 8 \\ 3x + y = 3 \end{cases}$$

$x = \boxed{2}$ and $y = \boxed{-3}$

Answer:

$$\begin{aligned} &\begin{cases} x - 2y = 8 \\ 3x + y = 3 \end{cases} \\ \sim &\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 3 & 1 & 3 \end{array}\right] \\ \sim &\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 7 & -21 \end{array}\right] \quad R_2 \leftarrow R_2 - 3R_1 \\ \sim &\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \end{array}\right] \quad R_2 \leftarrow \tfrac{1}{7}R_2 \\ \sim &\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \quad R_1 \leftarrow R_1 + 2R_2 \\ \sim &\begin{cases} x = 2 \\ y = -3 \end{cases} \end{aligned}$$

Ex 41: Use Gaussian elimination to solve:

$$\begin{cases} x + 3y = 5 \\ 2x + y = 5 \end{cases}$$

$x = \boxed{2}$ and $y = \boxed{1}$

Answer:

$$\begin{aligned} &\begin{cases} x + 3y = 5 \\ 2x + y = 5 \end{cases} \\ \sim &\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 1 & 5 \end{array}\right] \\ \sim &\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -5 & -5 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1 \\ \sim &\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right] \quad R_2 \leftarrow -\tfrac{1}{5}R_2 \\ \sim &\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - 3R_2 \\ \sim &\begin{cases} x = 2 \\ y = 1 \end{cases} \end{aligned}$$

Ex 42: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 3y = 13 \\ x + 4y = 14 \end{cases}$$

$x = \boxed{2}$ and $y = \boxed{3}$

Answer:

$$\begin{cases} 2x + 3y = 13 \\ x + 4y = 14 \end{cases}$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 13 \\ 1 & 4 & 14 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 13 \\ 0 & 5 & 15 \end{array}\right] \quad R_2 \leftarrow 2R_2 - R_1$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 13 \\ 0 & 1 & 3 \end{array}\right] \quad R_2 \leftarrow \frac{1}{5}R_2$$

$$\sim \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] \quad R_1 \leftarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \quad R_1 \leftarrow \frac{1}{2}R_1$$

$$\sim \begin{cases} x = 2 \\ y = 3 \end{cases}$$

Ex 43: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 3y = 7 \\ x - y = 1 \end{cases}$$

$x = \boxed{2}$ and $y = \boxed{1}$

Answer:

$$\begin{cases} 2x + 3y = 7 \\ x - y = 1 \end{cases}$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & -5 & -5 \end{array}\right] \quad R_2 \leftarrow 2R_2 - R_1$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 1 & 1 \end{array}\right] \quad R_2 \leftarrow -\frac{1}{5}R_2$$

$$\sim \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad R_1 \leftarrow \frac{1}{2}R_1$$

$$\sim \begin{cases} x = 2 \\ y = 1 \end{cases}$$

Ex 44: Use Gaussian elimination to solve:

$$\begin{cases} 2x + 5y = 16 \\ x + 2y = 7 \end{cases}$$

$x = \boxed{3}$ and $y = \boxed{2}$

Answer:

$$\begin{cases} 2x + 5y = 16 \\ x + 2y = 7 \end{cases}$$

$$\sim \left[\begin{array}{cc|c} 2 & 5 & 16 \\ 1 & 2 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 5 & 16 \\ 0 & -1 & -2 \end{array}\right] \quad R_2 \leftarrow 2R_2 - R_1$$

$$\sim \left[\begin{array}{cc|c} 2 & 5 & 16 \\ 0 & 1 & 2 \end{array}\right] \quad R_2 \leftarrow -R_2$$

$$\sim \left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right] \quad R_1 \leftarrow R_1 - 5R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \quad R_1 \leftarrow \frac{1}{2}R_1$$

$$\sim \begin{cases} x = 3 \\ y = 2 \end{cases}$$

E.2 SOLVING 3×3 SYSTEMS OF LINEAR EQUATIONS: LEVEL 1

Ex 45: Use Gaussian elimination to solve:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ x + 2y + 3z = 14 \end{cases}$$

$x = \boxed{1}$, $y = \boxed{2}$, $z = \boxed{3}$

Answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & 1 & 2 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 1 & 2 & 8 \end{array}\right] \quad R_2 \leftarrow -R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_3 \leftarrow -R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$x = 1,\ y = 2,\ z = 3.$

Ex 46: Use Gaussian elimination to solve:

$$\begin{cases} x + 2y + z = 7 \\ 2x - y + z = 6 \\ 3x + y - 2z = 1 \end{cases}$$

$x = \boxed{2}$, $y = \boxed{1}$, $z = \boxed{3}$

Answer:

$$\begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 2 & -1 & 1 & | & 6 \\ 3 & 1 & -2 & | & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & -5 & -1 & | & -8 \\ 0 & -5 & -5 & | & -20 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & -5 & -1 & | & -8 \\ 0 & 0 & -4 & | & -12 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & -5 & -1 & | & -8 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_3 \leftarrow -\frac{1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & -5 & 0 & | & -5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_2 \leftarrow -\frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_1 \leftarrow R_1 - 2R_2 - R_3$$

$x = 2, y = 1, z = 3$.

Ex 47: Use Gaussian elimination to solve:

$$\begin{cases} x + y + 2z = 9 \\ 2x + 3y + z = 11 \\ -x + y + z = 4 \end{cases}$$

$x = \boxed{1}$, $y = \boxed{2}$, $z = \boxed{3}$

Answer:

$$\begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 2 & 3 & 1 & | & 11 \\ -1 & 1 & 1 & | & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3 & | & -7 \\ 0 & 2 & 3 & | & 13 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3 & | & -7 \\ 0 & 0 & 9 & | & 27 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3 & | & -7 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_3 \leftarrow \frac{1}{9}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 + 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \quad R_1 \leftarrow R_1 - R_2$$

$x = 1, y = 2, z = 3$.

Ex 48: Use Gaussian elimination to solve:

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 3 \\ x + 2y + 3z = 13 \end{cases}$$

$x = \boxed{1}$, $y = \boxed{3}$, $z = \boxed{2}$

Answer:

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 2 & 1 & -1 & | & 3 \\ 1 & 2 & 3 & | & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -1 & -3 & | & -9 \\ 0 & 1 & 2 & | & 7 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -1 & -3 & | & -9 \\ 0 & 0 & -1 & | & -2 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & -1 & -3 & | & -9 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \quad R_3 \leftarrow -R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & | & 4 \\ 0 & -1 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \quad R_2 \leftarrow -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \quad R_1 \leftarrow R_1 - R_2$$

$x = 1, y = 3, z = 2$.

F ANALYZING SOLUTIONS FROM ROW-ECHELON FORM

F.1 ANALYZING 2X2 SYSTEMS WITH PARAMETERS

Ex 49: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} x + ky = 3 \\ 2x + 4y = 1 \end{cases}$$

Determine the values of k for which the system has:

1. a unique solution.
2. no solution.
3. infinitely many solutions.

For the case where a unique solution exists, find the expressions for x and y in terms of k.

Answer: We begin by writing the system as an augmented matrix and performing Gaussian elimination.

$$\begin{bmatrix} 1 & k & | & 3 \\ 2 & 4 & | & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & k & | & 3 \\ 0 & 4-2k & | & -5 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

Now we analyze the second row, which corresponds to the equation $(4-2k)y = -5$.

1. **Unique Solution:**
   A unique solution exists if we can uniquely solve for y. This is possible if the coefficient of y is not zero.
   $$4 - 2k \neq 0 \implies 2k \neq 4 \implies \boldsymbol{k \neq 2}$$
2. **No Solution:**
   If the coefficient of y is zero, the system has no unique

solution. Let's check the case where $k = 2$. The second row becomes:

$$[0 \quad 0 \quad | \quad -5]$$

This corresponds to the equation $0y = -5$, or $0 = -5$. This is a contradiction. Therefore, when $\boldsymbol{k = 2}$, the system has **no solution**.

3. **Infinitely Many Solutions:**
   For infinitely many solutions, the last row must be of the form '[0 0 | 0]'. In our case, the last element is always -5. It is therefore impossible for this system to have infinitely many solutions.

Solution in terms of k (for $k \neq 2$):
From the row-echelon form, we use back-substitution.
From R_2: $(4-2k)y = -5 \implies \boldsymbol{y = \frac{-5}{4-2k} = \frac{5}{2k-4}}$.
From R_1: $x + ky = 3 \implies x = 3 - ky$.

$$\begin{aligned} x &= 3 - k\left(\frac{5}{2k-4}\right) \\ &= \frac{3(2k-4)-5k}{2k-4} \\ &= \frac{6k-12-5k}{2k-4} \\ &= \boldsymbol{\frac{k-12}{2k-4}} \end{aligned}$$

Ex 50: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} 2x - y = 4 \\ -6x + 3y = k \end{cases}$$

Determine the values of k for which the system has:

1. a unique solution.
2. no solution.
3. infinitely many solutions.

For the case where infinitely many solutions exist, find the general solution in parametric form.

Answer: We write the system as an augmented matrix and perform Gaussian elimination.

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ -6 & 3 & k \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -1 & 4 \\ 0 & 0 & k+12 \end{array}\right] \quad {\scriptstyle R_2 \leftarrow R_2 + 3R_1}$$

The second row corresponds to the equation $0y = k + 12$.

1. **Unique Solution:**
   A unique solution exists only if the coefficient of y in the second row is non-zero. Here, the coefficient is always 0. Therefore, this system can **never** have a unique solution.

2. **No Solution:**
   The system has no solution if the second row leads to a contradiction. The equation is $0 = k + 12$. This is a contradiction if $k + 12 \neq 0$. Therefore, the system has no solution when $\boldsymbol{k \neq -12}$.

3. **Infinitely Many Solutions:**
   The system has infinitely many solutions if the second row becomes the identity $0 = 0$. This occurs when:

$$k + 12 = 0 \implies \boldsymbol{k = -12}$$

Parametric Solution (for $k = -12$):
When $k = -12$, the system reduces to a single equation: $2x - y = 4$.
To find the parametric solution, let $x = t$, where $t \in \mathbb{R}$.

$$2t - y = 4 \implies \boldsymbol{y = 2t - 4}$$

The general solution is $\boldsymbol{(x, y) = (t, 2t-4)}$.

F.2 ANALYZING 3X3 SYSTEMS WITH PARAMETERS

Ex 51: Consider the following system of linear equations, where $k \in \mathbb{R}$:

$$\begin{cases} x + y + z = 3 \\ x + 2y - z = 4 \\ 2x + 3y + kz = 8 \end{cases}$$

1. Find the value of k for which the system does not have a unique solution.
2. For this value of k, determine if the system has no solution or infinitely many solutions.
3. Describe the geometric interpretation of the system for this value of k.
4. Given that $k = 1$, find the unique solution to the system.

Answer:

1. We write the system as an augmented matrix and perform Gaussian elimination.

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & k & 8 \end{array}\right] \\ \sim &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & k-2 & 2 \end{array}\right] \quad {\scriptstyle R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - 2R_1} \\ \sim &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & k & 1 \end{array}\right] \quad {\scriptstyle R_3 \leftarrow R_3 - R_2} \end{aligned}$$

   The system does not have a unique solution when the coefficient of the last variable in the last row is zero. This occurs when $\boldsymbol{k = 0}$.

2. When $k = 0$, the last row of the matrix becomes '[0 0 0 | 1]'. This corresponds to the equation $0 = 1$, which is a contradiction. Therefore, the system has **no solution**.

3. The three equations represent three planes in 3D space. Since the system has no solution and no two planes are parallel, the planes intersect in pairs to form a triangular prism.

4. Given $k = 1$, we continue from the row-echelon form found in part (a).

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \\ \sim &\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad {\scriptstyle R_1 \leftarrow R_1 - R_3,\ R_2 \leftarrow R_2 + 2R_3} \\ \sim &\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad {\scriptstyle R_1 \leftarrow R_1 - R_2} \end{aligned}$$

   The unique solution is $x = -1, y = 3, z = 1$.

Ex 52: Consider the following system of linear equations, where $a \in \mathbb{R}$:

$$\begin{cases} x + 2y - z = 1 \\ 2x + 5y + z = 5 \\ x + 3y + 2z = a \end{cases}$$

1. Find the value of a for which the system has infinitely many solutions.

2. For this value of a, find the general solution to the system of equations.

3. Describe the geometric interpretation of the system for this value of a.

Answer:

1. We write the system as an augmented matrix and perform Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 1 & 5 \\ 1 & 3 & 2 & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 3 & a-1 \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & a-4 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

For the system to have infinitely many solutions, the last row must be all zeros. This means the final entry must also be zero.

$$a - 4 = 0 \implies \boldsymbol{a = 4}$$

2. With $a = 4$, the row-echelon form is $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The last row $0 = 0$ is redundant. We have two equations for three variables. Let $z = t$, where $t \in \mathbb{R}$. From R_2: $y + 3z = 3 \implies y + 3t = 3 \implies \boldsymbol{y = 3 - 3t}$. From R_1: $x + 2y - z = 1 \implies x + 2(3 - 3t) - t = 1 \implies x + 6 - 6t - t = 1 \implies x = -5 + 7t$. The general solution is $\boldsymbol{x = -5 + 7t, y = 3 - 3t, z = t}$.

3. The three equations represent three planes in 3D space. Since the system has infinitely many solutions, the three planes intersect along a common line.

Ex 53: Consider the following system of linear equations, where $k, m \in \mathbb{R}$:

$$\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 4 \\ 2x + 3y + kz = m \end{cases}$$

1. Find the value of k for which the system does not have a unique solution.

2. For this value of k, find the value of m for which the system has infinitely many solutions.

3. For the values of k and m found in parts (a) and (b), find the general solution to the system.

Answer:

1. We write the system as an augmented matrix and perform Gaussian elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & k & m \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & k-2 & m-2 \end{array}\right] \begin{array}{l} \\ R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & k-4 & m-5 \end{array}\right] \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

The system does not have a unique solution when the leading coefficient in the third row is zero. This occurs when $k - 4 = 0$, so $\boldsymbol{k = 4}$.

2. For the system to have infinitely many solutions, the last row of the row-echelon matrix must be entirely zeros, representing the identity $0 = 0$. With $k = 4$, the last row is '[0 0 0 | m-5]'. For this row to be all zeros, we need:

$$m - 5 = 0 \implies \boldsymbol{m = 5}$$

(If $m \neq 5$, the system would have no solution).

3. With $k = 4$ and $m = 5$, the system reduces to:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{cases} x + y + z = 1 \\ y + 2z = 3 \end{cases}$$

Let $z = t$, where $t \in \mathbb{R}$.
From the second equation: $y + 2t = 3 \implies \boldsymbol{y = 3 - 2t}$.
Substitute into the first equation: $x + (3 - 2t) + t = 1 \implies x + 3 - t = 1 \implies \boldsymbol{x = t - 2}$.
The general solution is $\boldsymbol{(x, y, z) = (t - 2, 3 - 2t, t)}$.