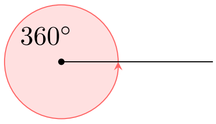
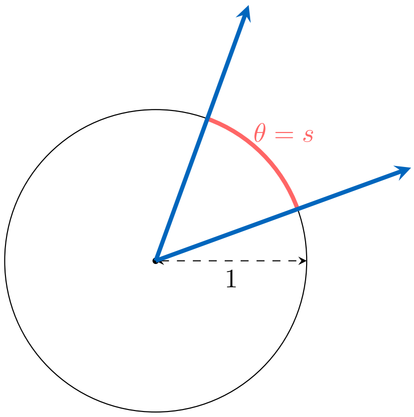
TRIGONOMETRY

- The measure of an angle is the fraction of a complete turn. We have seen previously that a full revolution makes an angle of 360°.



  Although a full revolution could just as well be 100°, or 120°, or some other number of degrees. This makes degrees a rather artificial measure of angles.

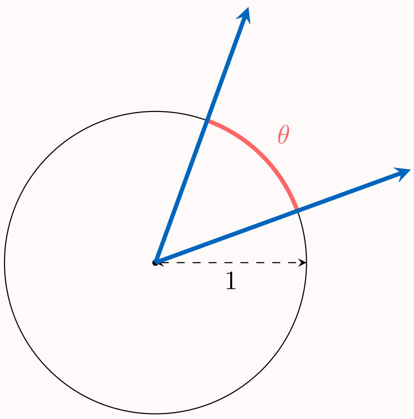
- An angle can be equal to the length of the arc formed by an angle with the circle of radius 1.



A RADIAN MEASURE

Definition Radian Measure

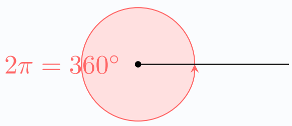
The **radian measure**, θ, of an angle is defined as the length of the arc subtended between the two rays of the angle on a circle of radius 1.



The radian unit is often omitted with 1 rad = 1.

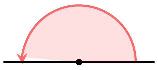
Proposition Angle of Complete Turn

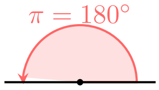
The radian measure of the revolution angle is 2π.



Proof

The circumference of the unit circle is 2π.
By definition, the revolution angle is equal to the arc length, which is 2π.

Ex: Find the angle of the straight angle 

Answer: The straight angle is half the revolution angle. So the angle is $\frac{2\pi}{2} = \pi$. 

Proposition Proportionality of Radians and Degrees

Radians and degrees are in proportion.

Method Converting Between Degrees and Radians

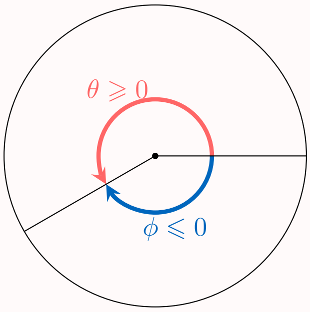
- To convert from degrees to radians, multiply degrees by $\frac{\pi}{180}$
- To convert from radians to degrees, multiply radians by $\frac{180}{\pi}$

Ex: Convert 60° to radians.

Answer:
$$\begin{aligned} 60° &= 60° \times \frac{\pi}{180°} \\ &= \frac{\pi}{3} \end{aligned}$$

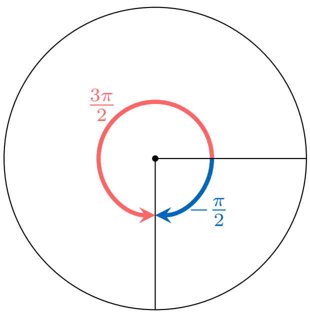
Definition Positive and Negative Angles

- A **positive angle measure** represents a **counterclockwise rotation**.
- A **negative angle measure** represents a **clockwise rotation**.

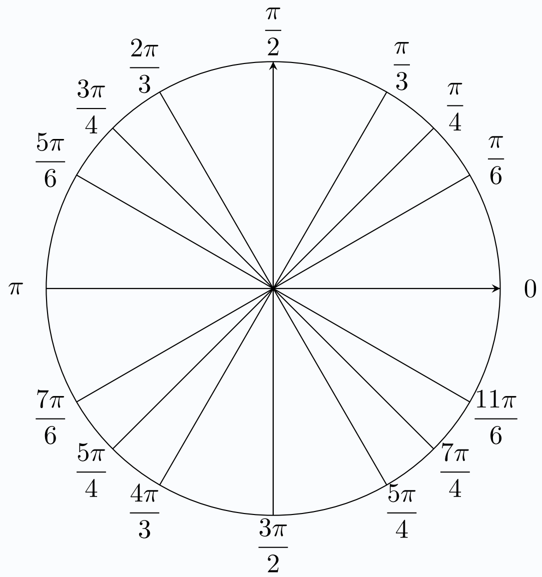


Ex: Draw the angle $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$

Answer:



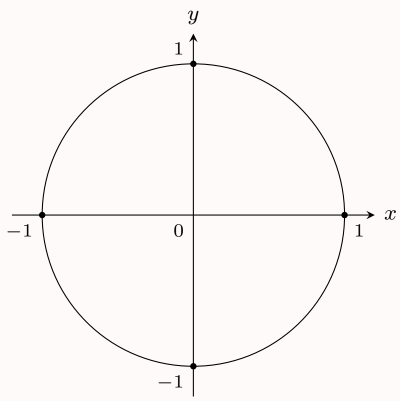
Proposition Reference Angles on the Unit Circle



B TRIGONOMETRIC RATIOS AND UNIT CIRCLE

Definition Unit circle

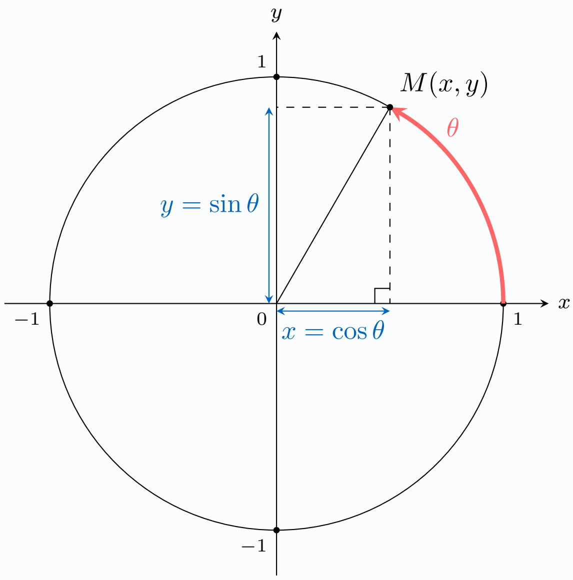
The **unit circle** is a circle with a radius of 1 centered at the origin.



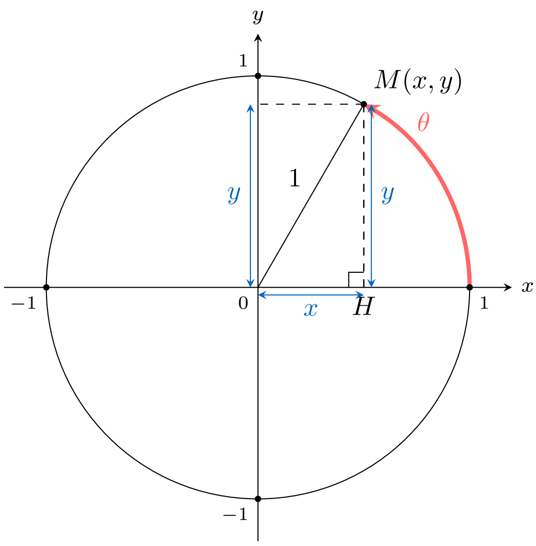
Proposition Relation between Angle and Coordinates

For $M(x, y)$ the point on the unit circle at the angle θ, then

- $\cos\theta$ is the x-coordinate of M: $\cos\theta = x$
- $\sin\theta$ is the y-coordinate of M: $\sin\theta = y$



Proof



Using right-angled triangle trigonometry in the right triangle OHM:

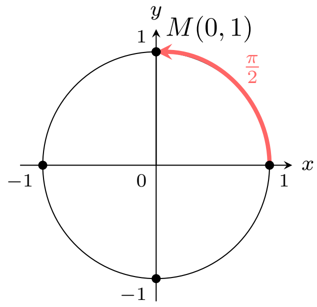
$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ &= \frac{OH}{OM}\\ &= \frac{x}{1}\\ &= x\end{aligned}$$

and

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ &= \frac{HM}{OM}\\ &= \frac{y}{1}\\ &= y\end{aligned}$$

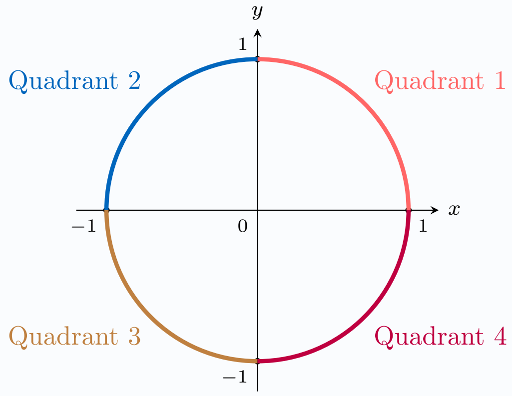
Ex: Find the values $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

Answer: On the unit circle, the point corresponding to the angle $\frac{\pi}{2}$ has coordinates (0,1):



$\cos\left(\frac{\pi}{2}\right) = 0 \quad x\text{-coordinate}$
$\sin\left(\frac{\pi}{2}\right) = 1 \quad y\text{-coordinate}$

Proposition **Sign of of Sine and Cosine**



Quadrant	$\cos\theta$	$\sin\theta$
1	$+$	$+$
2	$-$	$+$
3	$-$	$-$
4	$+$	$-$

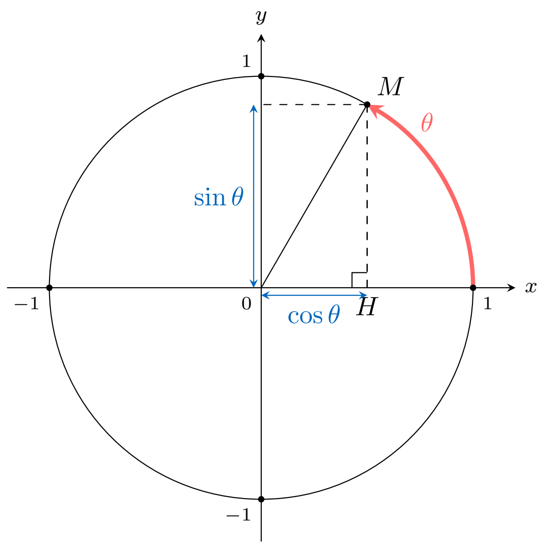
C TRIGONOMETRIC PROPERTIES

Proposition **Pythagorean Relation**

$$\cos^2\theta + \sin^2\theta = 1$$

Proof

Let $M(\cos\theta, \sin\theta)$ the point on the unit circle at the angle θ.



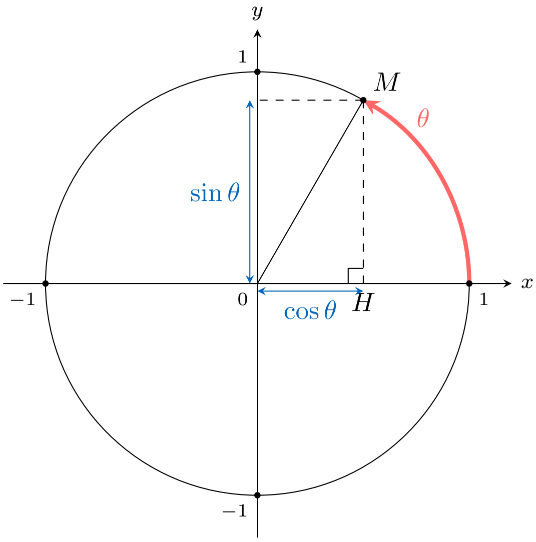
By Pythagorean theorem in the right triangle OHM:

$$OH^2 + HM^2 = OM^2$$
$$(\cos\theta)^2 + (\sin\theta)^2 = 1^2$$
$$\cos^2\theta + \sin^2\theta = 1$$

Proposition Maximum and Minimum of Trigonometric Ratios

$$-1 \leqslant \cos\theta \leqslant 1 \quad \text{and} \quad -1 \leqslant \sin\theta \leqslant 1$$

Proof



- Geometric proof:
  The length OH lies in $[-1, 1]$. As $OH = \cos\theta$, $-1 \leqslant \cos\theta \leqslant 1$.

- Analytical proof:

$$\begin{aligned}
0 &\leqslant \sin^2\theta && \text{(square number is positive)}\\
\cos^2\theta &\leqslant \cos^2\theta + \sin^2\theta && \text{(adding } \cos^2\theta \text{ to both sides)}\\
\cos^2\theta &\leqslant 1 && (\cos^2\theta + \sin^2\theta = 1)\\
-1 \leqslant \cos\theta &\leqslant 1 && \text{(taking square roots, as } \sqrt{\cos^2\theta} = |\cos\theta| \leqslant 1)
\end{aligned}$$

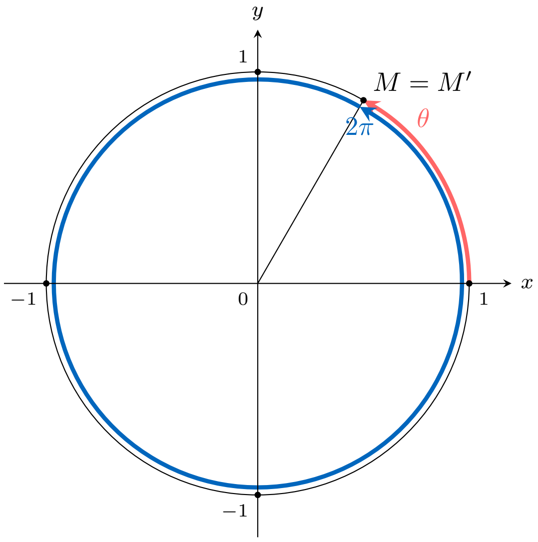
Proposition Periodicity of Trigonometric Ratios

$$\cos(\theta + 2k\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2k\pi) = \sin\theta$$

where k is any integer.

Proof

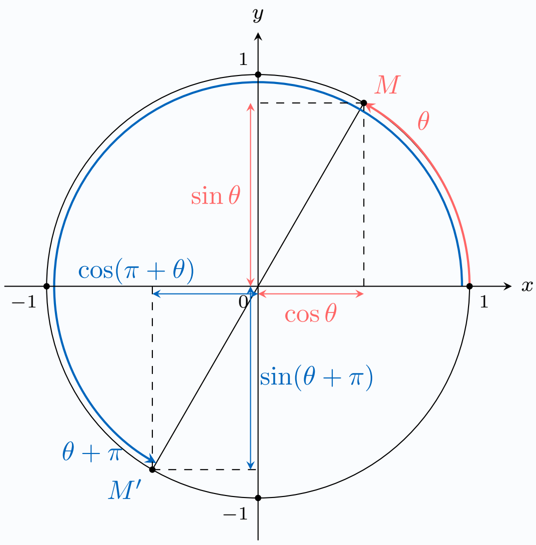
Let $M(\cos\theta, \sin\theta)$ the point on the unit circle at the angle θ.
Let $M'(\cos(\theta + 2\pi), \sin(\theta + 2\pi))$ the point on the unit circle at the angle $\theta + 2\pi$.



2π is a full revolution. So, the position on the unit circle is the same: $M' = M$.
Thus, $\cos(\theta + 2\pi) = \cos\theta$ and $\sin(\theta + 2\pi) = \sin\theta$.
The proof extends to any multiple of 2π, i.e., for integer k.

Proposition Add π to Trigonometric Ratios

$$\sin(\pi + \theta) = -\sin\theta$$
$$\cos(\pi + \theta) = -\cos\theta$$



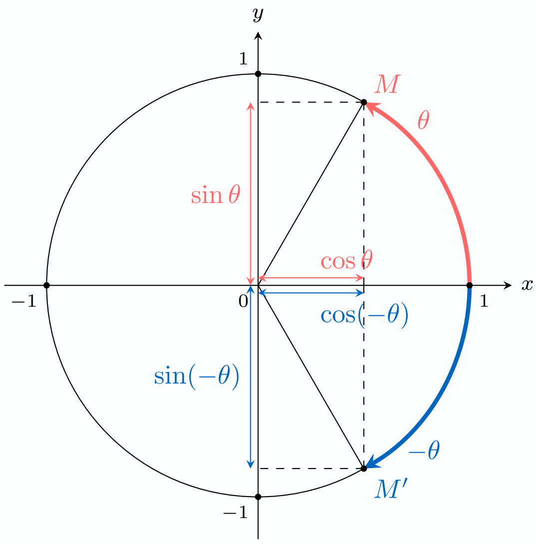
Proof

Let θ an angle.
Let $M(\cos\theta, \sin\theta)$ the point on the unit circle at the angle θ.
Let $M'(\cos(\pi+\theta), \sin(\pi+\theta))$ the point on the unit circle at the angle $\pi + \theta$.
As the rotation of angle π is the point reflection through the origin O, the coordinates of M' are the opposites of the coordinates of M.
So $\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

Proposition Opposite of Trigonometric Ratios

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$



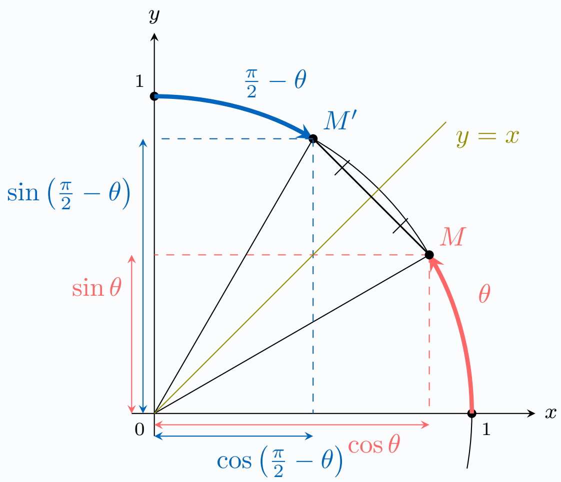
Proof

Let $M(\cos\theta, \sin\theta)$ the point on the unit circle at the angle θ.
Let $M'(\cos(-\theta), \sin(-\theta))$ the point on the unit circle at the angle $-\theta$.
As M' is the reflection over the x-axis of M, the x-coordinate of M' is the same as M, and the y-coordinate of M' is the opposite of M's y-coordinate.
So $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

Proposition **Identities with $\frac{\pi}{2} - x$**

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$



Proof

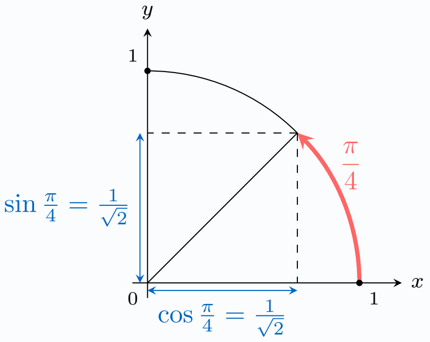
Let θ an angle.
Let $M(\cos\theta, \sin\theta)$ the point on the unit circle at the angle θ.
Let $M'(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$ the point on the unit circle at the angle $\frac{\pi}{2} - \theta$.
As the point M is the reflection over the line $y = x$ of M', the x-coordinate of M is the y-coordinate of M', and the y-coordinate of M is the x-coordinate of M'.
So $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

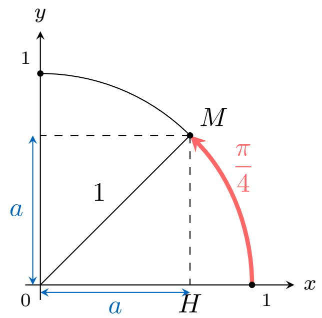
D MULTIPLES OF $\frac{\pi}{4}$

Proposition **Coordinates of Angle $\frac{\pi}{4}$**

$$\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$



Proof



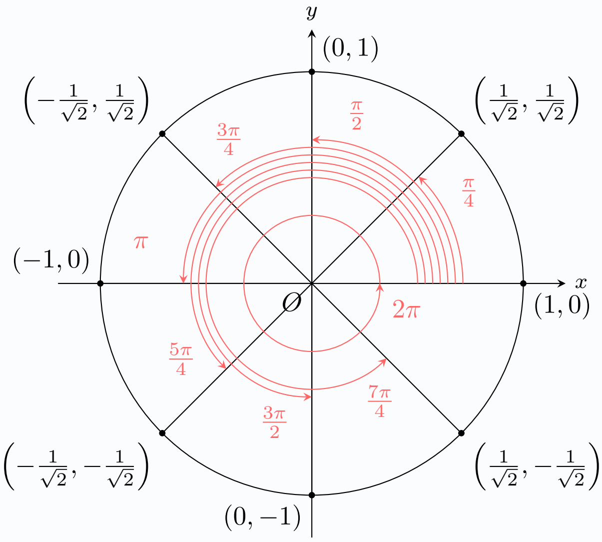
As the sum of angles in a triangle is π, $\angle OMH=\pi-\frac{\pi}{4}-\frac{\pi}{2}=\frac{\pi}{4}$.
As $\angle OMH=\angle MOH$, the triangle OHM is isosceles.
Let $a=OH=HM$.

$$\begin{aligned} a^2+a^2&=1^2 \quad \text{Pythagorean theorem for the right triangle } OHM\\ 2a^2&=1\\ a^2&=\frac{1}{2}\\ a&=\frac{1}{\sqrt{2}} \quad \text{as } a\geqslant 0 \end{aligned}$$

So $M\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.
As $\cos\theta$ is the x-coordinate of M and $\sin\theta$ is the y-coordinate of M,

$$\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

Proposition **Multiples of $\frac{\pi}{4}$**



Proof

The coordinates of each point are found by reflection symmetries over the axes or the origin.

The signs of the coordinates are determined by the quadrant in which the angle lies.

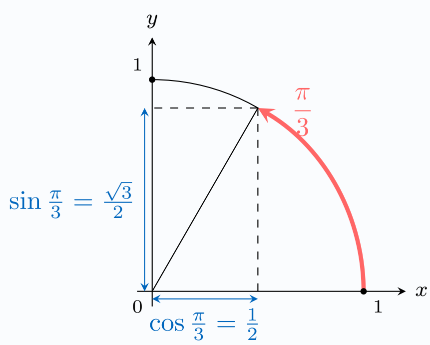
Ex: Find $\cos \frac{3\pi}{4}$

Answer: $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

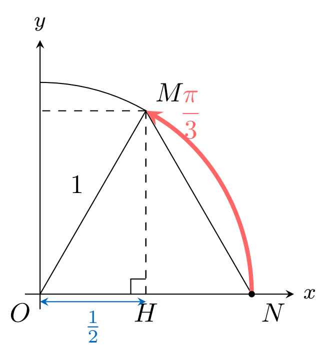
E MULTIPLES OF $\frac{\pi}{6}$

Proposition **Coordinates of Angle $\frac{\pi}{3}$**

$$\cos \frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$



Proof



Let $\angle MON = \frac{\pi}{3}$.
As $ON = OM = 1$, the triangle OMN is isosceles. So $\angle MON = \angle MNO = \frac{\pi}{3}$.
As the sum of angles in a triangle is π, $\angle OMN = \frac{\pi}{3}$.
So the triangle OMN is equilateral.

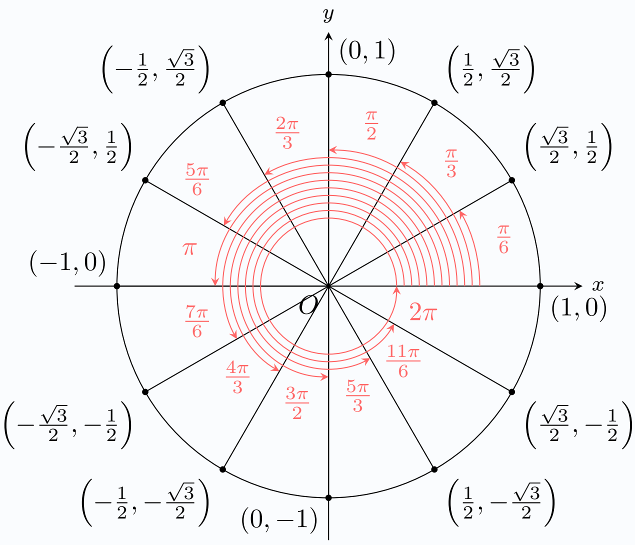
The altitude MH bisects the base ON.
So $OH = \frac{1}{2}$.

$$\begin{aligned} OH^2 + HM^2 &= OM^2 \quad \text{Pythagorean theorem for the right triangle } OHM \\ \left(\frac{1}{2}\right)^2 + HM^2 &= 1 \\ HM^2 &= \frac{3}{4} \\ HM &= \frac{\sqrt{3}}{2} \quad \text{as } HM \geqslant 0 \end{aligned}$$

As $\cos\theta$ is the x-coordinate of M and $\sin\theta$ is the y-coordinate of M:

$$\cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Proposition Multiples of $\frac{\pi}{6}$

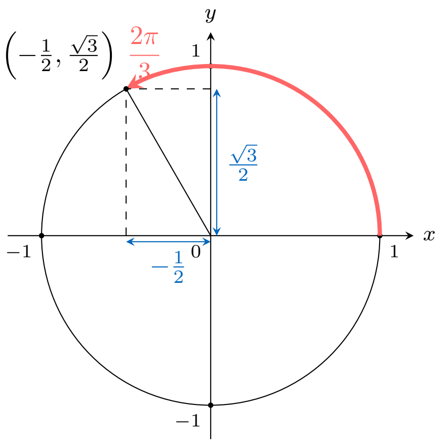


Proof

The coordinates of each point are found by reflection symmetries over the axes or the origin.

Ex: Find $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$

Answer:



$\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$