TRIGONOMETRY

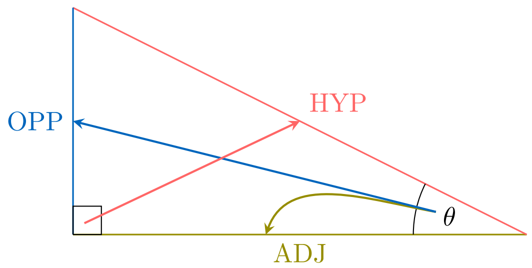
Trigonometry is a branch of mathematics that explores the relationships between the side lengths and angles of triangles, particularly right-angled triangles. It has wide applications in fields such as science, engineering, astronomy, and video game development. The foundation of trigonometry rests on three primary ratios: sine, cosine, and tangent.

A RIGHT-ANGLED TRIANGLE

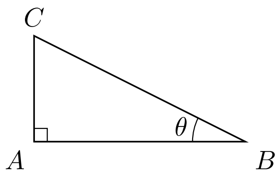
Definition Right-Angled Triangle

A **right-angled triangle** is a triangle with one angle equal to $90°$. For a given angle θ (other than the right angle), we define:

- **Hypotenuse (HYP)**: The longest side, opposite the right angle.
- **Adjacent Side (ADJ)**: The side adjacent to the angle θ, forming one side of the angle.
- **Opposite Side (OPP)**: The side opposite the angle θ.



Ex: In the triangle below, identify the hypotenuse, the adjacent side, and the opposite side relative to the angle θ.



Answer:

- Hypotenuse: $\overline{BC}$
- Adjacent side: $\overline{AC}$
- Opposite side: $\overline{AB}$

B TRIGONOMETRIC FUNCTIONS

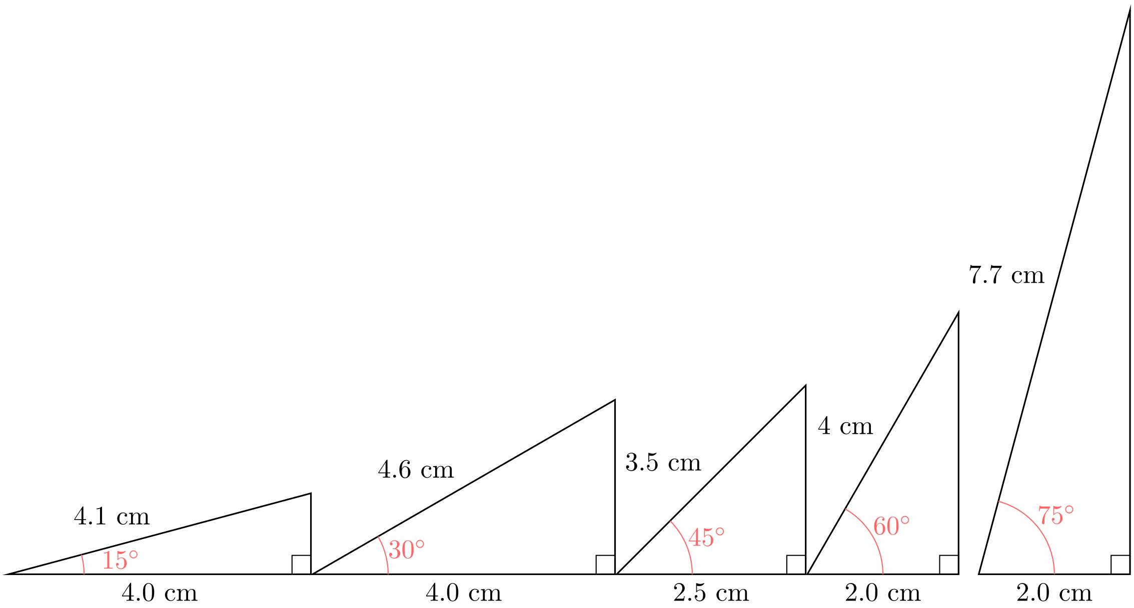
Discover:

1. Using a protractor and ruler, draw right-angled triangles with angles θ of $15°$, $30°$, $45°$, $60°$, and $75°$. Measure the lengths of the hypotenuse and the adjacent side to the nearest millimeter.
2. Complete the following table with the ratio $\frac{\text{ADJ}}{\text{HYP}}$ for each angle:

θ	$15°$	$30°$	$45°$	$60°$	$75°$
$\dfrac{\text{ADJ}}{\text{HYP}}$					

Answer:

1. Examples of right-angled triangles with the specified angles:



2. The ratio $\frac{\text{ADJ}}{\text{HYP}}$ should yield consistent values for each angle, approximately:

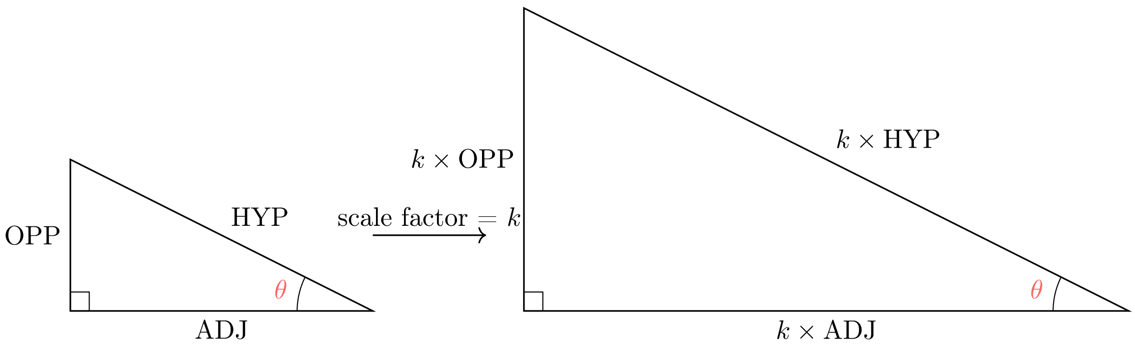
θ	15°	30°	45°	60°	75°
$\frac{\text{ADJ}}{\text{HYP}}$	$\frac{4.0}{4.1} \approx 0.98$	$\frac{4.0}{4.6} \approx 0.87$	$\frac{2.5}{3.5} \approx 0.71$	$\frac{2.0}{4.0} = 0.50$	$\frac{2.0}{7.7} \approx 0.26$

Proposition Trigonometric Ratios

For any two right-angled triangles with the same angle θ, the ratios $\frac{\text{OPP}}{\text{HYP}}$, $\frac{\text{ADJ}}{\text{HYP}}$, and $\frac{\text{OPP}}{\text{ADJ}}$ are constant.

Proof

Consider two right-angled triangles with the same angle θ. Since their angles are equal, the triangles are similar. Let k be the scale factor:



Ratios of first triangle: $\frac{\text{ADJ}}{\text{HYP}}$ $\frac{\text{OPP}}{\text{HYP}}$ $\frac{\text{OPP}}{\text{ADJ}}$

Ratios of second triangle: $\frac{k \times \text{ADJ}}{k \times \text{HYP}} = \frac{\text{ADJ}}{\text{HYP}}$ $\frac{k \times \text{OPP}}{k \times \text{HYP}} = \frac{\text{OPP}}{\text{HYP}}$ $\frac{k \times \text{OPP}}{k \times \text{ADJ}} = \frac{\text{OPP}}{\text{ADJ}}$

The ratios remain constant due to the similarity of the triangles.

Due to the constant nature of the ratio $\frac{\text{ADJ}}{\text{HYP}}$ for any right-angled triangle with angle θ, we define the **cosine** function, denoted cos, where the input is θ and the output is $\frac{\text{ADJ}}{\text{HYP}}$. For example:

θ	15°	30°	45°	60°	75°
$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}$	0.97	0.87	0.71	0.50	0.26

For instance, $\cos(45°) \approx 0.71$.

Definition Trigonometric Functions

In a right-angled triangle with angle θ:

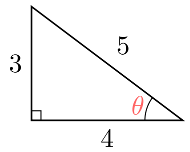
$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}}, \quad \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}, \quad \tan(\theta) = \frac{\text{OPP}}{\text{ADJ}}$$

The mnemonic **SOH-CAH-TOA** helps recall the definitions of sine, cosine, and tangent:

- **S**ine = **O**pposite ÷ **H**ypotenuse
- **C**osine = **A**djacent ÷ **H**ypotenuse
- **T**angent = **O**pposite ÷ **A**djacent

To aid memorization, you can listen this song: https://www.youtube.com/watch?v=PIWJo5uK3Fo.

Ex: In the triangle below, find $\cos\theta$, $\sin\theta$, and $\tan\theta$.



Answer: Relative to θ:

- Hypotenuse: $BC = 5$
- Adjacent side: $AB = 4$
- Opposite side: $AC = 3$

$$\begin{aligned} \cos\theta &= \frac{\text{ADJ}}{\text{HYP}} = \frac{4}{5} \\ \sin\theta &= \frac{\text{OPP}}{\text{HYP}} = \frac{3}{5} \\ \tan\theta &= \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{4} \end{aligned}$$

Proposition Tangent Formula

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

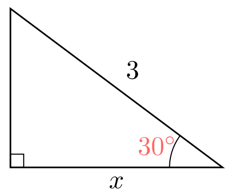
Proof

$$\begin{aligned} \frac{\sin\theta}{\cos\theta} &= \frac{\frac{\text{OPP}}{\text{HYP}}}{\frac{\text{ADJ}}{\text{HYP}}} \\ &= \frac{\text{OPP}}{\text{HYP}} \times \frac{\text{HYP}}{\text{ADJ}} \\ &= \frac{\text{OPP}}{\text{ADJ}} \\ &= \tan\theta \end{aligned}$$

Method Using Calculator

Trigonometric ratios for any angle can be calculated using a calculator in degree mode. Ensure your calculator is set to "degrees" before performing calculations.

Ex: In the triangle below, find x.



Answer:

$$\begin{aligned}\cos\theta &= \frac{\text{ADJ}}{\text{HYP}}\\ \cos(30^\circ) &= \frac{x}{3}\\ x &= 3\times\cos(30^\circ)\\ x &\approx 3\times 0.866\\ x &\approx 2.6\,\text{cm}\end{aligned}$$

C INVERSE TRIGONOMETRIC FUNCTIONS

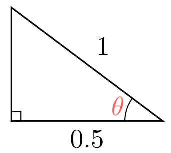
Trigonometric ratios can be used to find unknown angles in right-angled triangles when at least two side lengths are known.

Definition Inverse Trigonometric Functions

In a right-angled triangle with angle θ:

$$\theta=\cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right),\quad \theta=\sin^{-1}\left(\frac{\text{OPP}}{\text{HYP}}\right),\quad \theta=\tan^{-1}\left(\frac{\text{OPP}}{\text{ADJ}}\right)$$

Ex: In the triangle below, find θ.



Answer:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right)\\ &= \cos^{-1}\left(\frac{0.5}{1}\right)\\ &= \cos^{-1}(0.5)\\ &= 60^\circ\end{aligned}$$