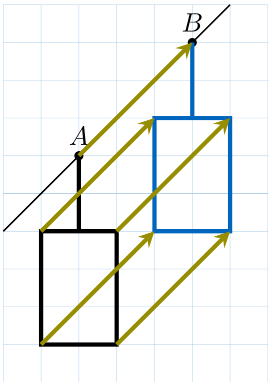
VECTORS

A DEFINITION

Moving a figure by **translation** means sliding the figure without rotating it. To describe this movement, we use an arrow $\vec{u}$ called a vector. To translate all points by a vector, we can use:

- the length of the arrow (**magnitude**) and the direction (orientation and sense).
- the x**-step** and y**-step** of the vector (how many units to the right/left and up/down).

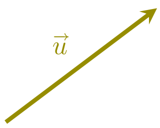


These two perspectives on a vector give us two definitions.

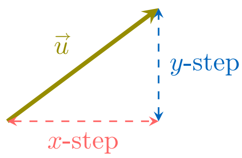
Definition Vector

A **vector** $\vec{u}$ can be defined:

1. **Geometrically**, as having a **direction** (including orientation and sense) and a **magnitude** (length).

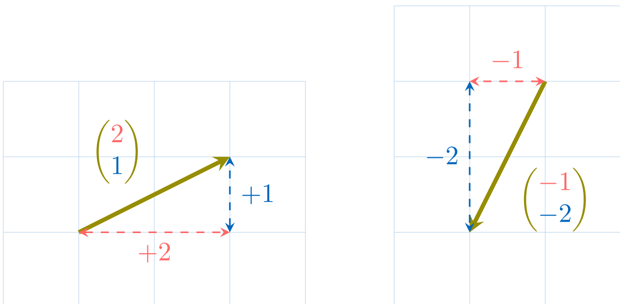


2. **Algebraically**, as an ordered pair of components $\begin{pmatrix} x-\text{step} \\ y-\text{step} \end{pmatrix}$.



- The x-step is positive if the movement is to the right and negative if to the left.
- The y-step is positive if the movement is upwards and negative if downwards.

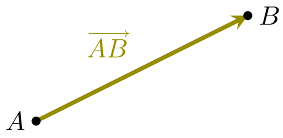
Ex:



B TWO POINT NOTATION

Definition Two Point Notation

The vector from A to B can be written as:



Proposition Components of $\overrightarrow{AB}$

Let $A(x_A, y_A)$ and $B(x_B, y_B)$ be two points. The components of the vector $\overrightarrow{AB}$ are:

$$\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$$

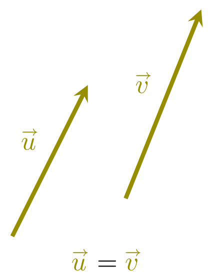
C EQUALITY BETWEEN VECTORS

Two vectors are said to be **equal** if they represent the same translation.

Definition Geometric Equality of Vectors

Two vectors are **equal** if they have:

- the same **direction** (parallel lines),
- the same **sense** (same orientation),
- and the same **magnitude** (same length).

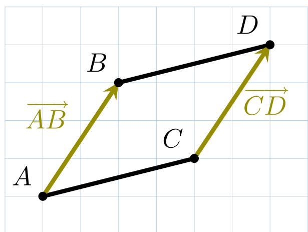


Definition Algebraic Equality of Vectors

Two vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ are equal if $x = x'$ and $y = y'$.

Proposition Parallelogram criterion

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if and only if $ABDC$ is a parallelogram (possibly flattened).



D ADDITION

Definition Vector Addition

The **sum** of two vectors $\vec{u}$ and $\vec{v}$, written $\vec{u}+\vec{v}$, is the vector corresponding to performing the translation by $\vec{u}$ followed by the translation by $\vec{v}$.

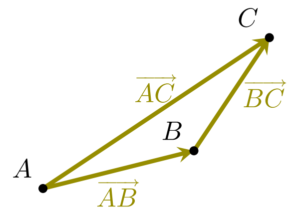
Definition Algebraic Vector Addition

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$$

Proposition Chasles' Relation

For any points A, B, and C,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



Having defined vector addition, we can now state that:

Definition Null Vector

The vector associated with the translation that transforms any point into itself is the **null vector**, denoted $\vec{0}$.

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$$

For any point A:

$$\overrightarrow{AA} = \vec{0}$$

E SUBTRACTION

Definition Negative Vector

The **negative vector** of $\vec{u}$ is the vector with the same direction, the same length, but the opposite sense (orientation). It is denoted by $-\vec{u}$.

Definition Algebraic Negative Vector

$$-\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Proposition Negative of $\overrightarrow{AB}$

The vector $\overrightarrow{BA}$ (from B to A) is the **negative** of the vector $\overrightarrow{AB}$ (from A to B):

$$\overrightarrow{BA} = -\overrightarrow{AB}$$

To subtract one vector from another, we simply add its negative (opposite).

Definition Subtraction of Vectors

The **subtraction of vectors** $\vec{u} - \vec{v}$ is defined as adding the negative of the second vector:

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

Ex: Calculate $\overrightarrow{AB} - \overrightarrow{CB}$.

Answer:

$$\begin{aligned} \overrightarrow{AB} - \overrightarrow{CB} &= \overrightarrow{AB} + (-\overrightarrow{CB}) \\ &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \overrightarrow{AC} \end{aligned}$$

Definition Algebraic Vector Subtraction

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x - x' \\ y - y' \end{pmatrix}$$

F SCALAR MULTIPLICATION

Definition Scalar Multiplication

Let k be a real number. The product $k\vec{v}$ is defined by:

$$k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$$

Remark

- $2\vec{v}$ has the same direction and sense as $\vec{v}$, and is twice as long.
- $-2\vec{v}$ has the same direction but the opposite sense, and is twice as long as $\vec{v}$.
- If $0 < k < 1$, the vector $k\vec{v}$ has the same direction and sense as $\vec{v}$, but is shorter.
- $0\vec{v} = \vec{0}$, the null vector.

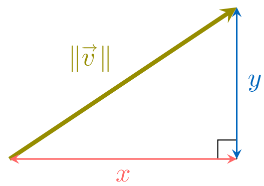
G MAGNITUDE OF A VECTOR

By the theorem of Pythagoras,

Proposition Magnitude of a Vector

If $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, the magnitude or length of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{x^2 + y^2}$$



Ex: Calculate the length of $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Answer:

$$\begin{aligned} \| \quad \vec{v} \quad \| &= \sqrt{4^3 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

Proposition Distance between Two Points

The **distance** between points $A(x_A, y_A)$ and $B(x_B, y_B)$ in the plane is the magnitude of the vector $\overrightarrow{AB}$:

$$AB = \|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Ex: Calculate the distance between the points $A(1,\ 2)$ and $B(5,\ 5)$.

Answer:

$$\begin{aligned} AB &= \| \quad \overrightarrow{AB} \quad \| \\ &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ &= \sqrt{(5-1)^2 + (5-2)^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

H COLINEARITY

Definition Colinear Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are said to be **colinear** if there exists a real number k such that:

$$\vec{u} = k\vec{v}$$

Remarks

- The components of two colinear vectors are proportional.
- The zero vector $\vec{0}$ is colinear with every vector $\vec{u}$, since $\vec{0} = 0 \cdot \vec{u}$.

Proposition Colinearity and Geometry

- Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are **parallel** if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are colinear.
- Three points A, B, and C are **collinear** (lie on the same straight line) if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are colinear.

Definition Determinant of Two Vectors

Let $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ be two vectors in the plane. The **determinant** of $\vec{u}$ and $\vec{v}$, denoted $\det(\vec{u}, \vec{v})$, is defined as:

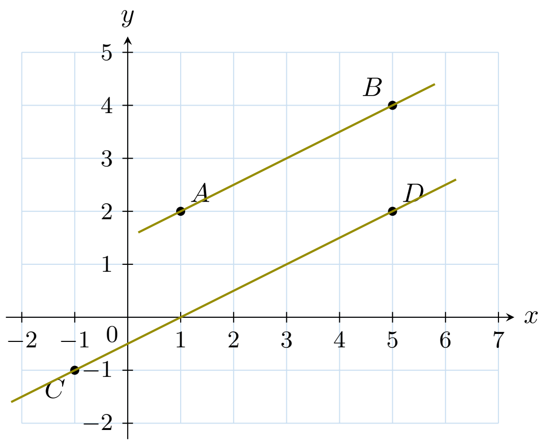
$$\det(\vec{u}, \vec{v}) = \begin{vmatrix} x & x' \\ y & y' \end{vmatrix} = x \cdot y' - y \cdot x'$$

Proposition Colinearity and Determinant

Two vectors $\vec{u}$ and $\vec{v}$ are **colinear** if and only if

$$\det(\vec{u}, \vec{v}) = 0$$

Ex: Let $A(1,\ 2)$, $B(5,\ 4)$, $C(-1,\ -1)$, and $D(5,\ 2)$.
Are the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ parallel?



Answer: First, compute the vectors:

$$\overrightarrow{AB} = \begin{pmatrix} 5-1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \overrightarrow{CD} = \begin{pmatrix} 5-(-1) \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Calculate the determinant:

$$\det(\overrightarrow{AB}, \overrightarrow{CD}) = \begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 6 \times 2 = 12 - 12 = 0$$

So the determinant is zero. Let's double-check the proportionality:

$$\frac{6}{4} = 1.5, \quad \frac{3}{2} = 1.5$$

Thus, $\overrightarrow{CD} = 1.5\overrightarrow{AB}$, so the vectors are colinear, and the lines are **parallel**.

Conclusion In this chapter, we have explored vectors as fundamental tools for describing movement and position in the plane, both geometrically and algebraically. Their strength lies in connecting algebraic operations with geometric intuition, enabling us to model and solve a wide variety of problems in mathematics and science.Vectors are not only essential in geometry but also play a key role in the modern world. In artificial intelligence and data science, for instance, almost all information—images, texts, and even relationships—are encoded as vectors. The landmark paper *Attention is All You Need* (Vaswani et al., 2017) underlines this fact: modern AI systems, such as OpenAI's ChatGPT or Grok, manipulate vast arrays of vectors to learn patterns and make predictions. As the authors write:

"An attention function can be described as mapping a query and a set of key-value pairs to an output, where the query, keys, values, and output are all ***vectors****."*

Understanding vectors, therefore, not only builds your geometric and algebraic fluency but also connects you to the mathematical foundations of today's digital and AI-driven world.