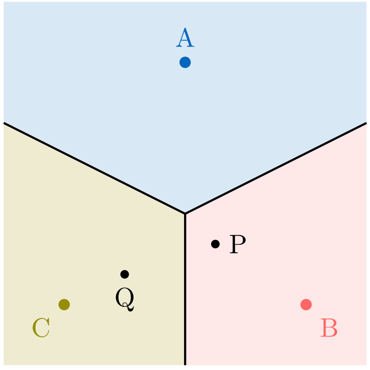
VORONOI DIAGRAMS

A DEFINITIONS

A.1 READING VORONOI DIAGRAMS

Ex 1: Consider the Voronoi diagram below for sites A, B, and C.



1. How many Voronoi regions are there?

$\boxed{3}$

2. Which site is closest to point P?

$\boxed{\textbf{B}}$

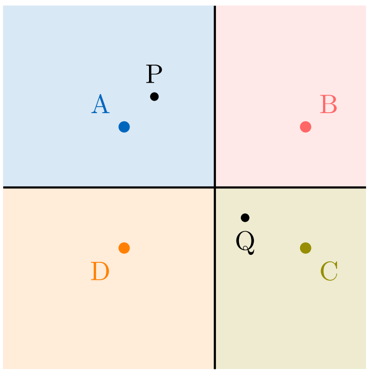
3. Which site is closest to point Q?

$\boxed{\textbf{C}}$

Answer:

1. There are 3 sites, so there are 3 regions.
2. Point P is inside the bottom-right region, which contains site B.
3. Point Q is inside the bottom-left region, which contains site C.

Ex 2: Consider the Voronoi diagram below for sites A, B, C, and D.



1. How many Voronoi regions are there?

$\boxed{4}$

2. Which site is closest to point P?

$\boxed{\textbf{A}}$

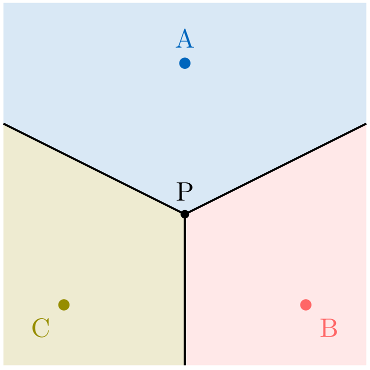
3. Which site is closest to point Q?

$\boxed{\textbf{C}}$

Answer:

1. There are 4 sites, so there are 4 regions.
2. Point P is inside the top-left region, which contains site A.
3. Point Q is inside the bottom-right region, which contains site C.

MCQ 3:



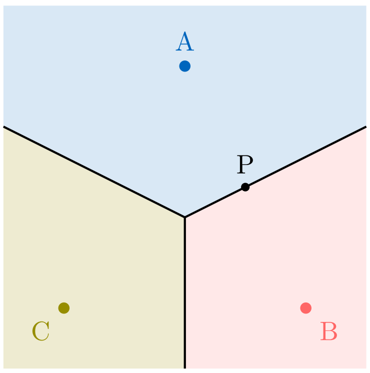
The point P is equidistant from the sites A, B, and C.

☒ True

☐ False

Answer: The point P is the **vertex** where the three edges meet. In a Voronoi diagram, a vertex is the intersection of the perpendicular bisectors of the sites. Therefore, it is the center of the circumcircle passing through these sites and is equidistant from them.

MCQ 4:



The point P is equidistant from sites A and B.

☒ True

☐ False

Answer: An edge in a Voronoi diagram is a segment of the **perpendicular bisector** of the two sites that share the boundary. By definition, any point on the perpendicular bisector is equidistant from the two endpoints (sites). Since P is on the edge between regions A and B, $PA = PB$.

MCQ 5: In a Voronoi diagram, what is true about a point lying exactly on an edge?

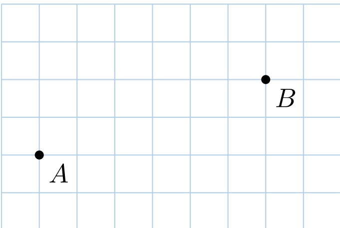
- ☐ It is closest to all sites in the diagram.
- ☒ It is equidistant from the two sites sharing that edge.
- ☐ It is the location of a site.

Answer: An edge is the perpendicular bisector of two sites, meaning any point on it is equidistant from those two sites.

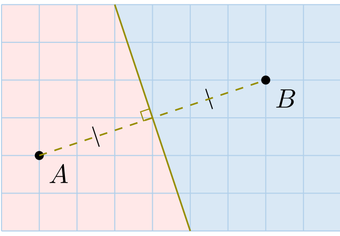
B CONSTRUCTING A VORONOI DIAGRAM

B.1 CONSTRUCTING VORONOI DIAGRAM FOR 2 SITES

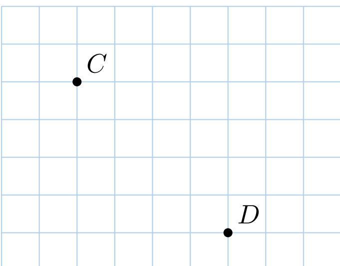
Ex 6: Using a ruler and a set square, draw the Voronoi diagram for two sites A and B.



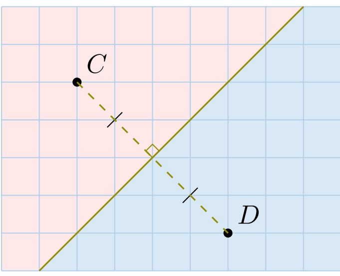
Answer: Draw the perpendicular bisector of the segment $[AB]$.



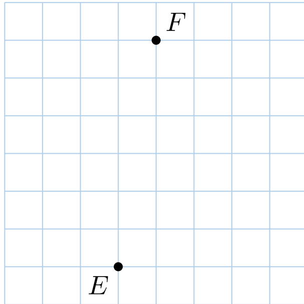
Ex 7: Using a ruler and a set square, draw the Voronoi diagram for two sites C and D.



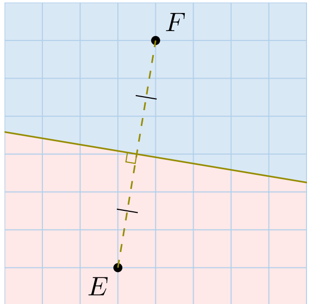
Answer: Draw the perpendicular bisector of the segment $[CD]$.



Ex 8: Using a ruler and a set square, draw the Voronoi diagram for two sites $E(3, 1)$ and $F(4, 7)$.



Answer: Draw the perpendicular bisector of the segment $[EF]$.



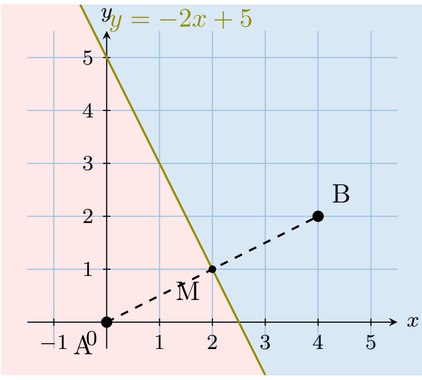
B.2 FINDING THE EQUATION OF THE PERPENDICULAR BISECTOR

Ex 9: Consider two sites $A(0, 0)$ and $B(4, 2)$.

1. Calculate the coordinates of the midpoint M of the segment $[AB]$.
2. Calculate the gradient (slope) of the segment $[AB]$.
3. Determine the gradient of the perpendicular bisector.
4. Find the equation of the perpendicular bisector in the form $y = mx + c$.

Answer:

1. Midpoint formula: $M\left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$.
$$M = \left(\frac{0+4}{2}, \frac{0+2}{2}\right) = (2, 1)$$
2. Gradient formula: $m = \frac{y_B-y_A}{x_B-x_A}$.
$$m = \frac{2-0}{4-0} = \frac{2}{4} = 0.5$$
3. Perpendicular gradient formula: $m_\perp = -\frac{1}{m}$.
$$m_\perp = -\frac{1}{0.5} = -2$$
4. Point-slope form: $y - y_M = m_\perp(x - x_M)$.
$$\begin{aligned} y - 1 &= -2(x-2) \\ y - 1 &= -2x + 4 \\ y &= -2x + 5 \end{aligned}$$

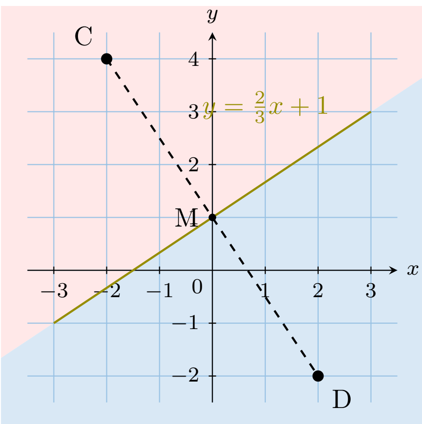


Ex 10: Consider two sites $C(-2,4)$ and $D(2,-2)$.

1. Calculate the coordinates of the midpoint M of the segment $[CD]$.
2. Calculate the gradient (slope) of the segment $[CD]$.
3. Determine the gradient of the perpendicular bisector.
4. Find the equation of the perpendicular bisector in the form $y = mx + c$.

Answer:

1. Midpoint formula: $M\left(\frac{x_C+x_D}{2}, \frac{y_C+y_D}{2}\right)$.
$$M = \left(\frac{-2+2}{2}, \frac{4+(-2)}{2}\right) = (0,1)$$
2. Gradient formula: $m = \frac{y_D-y_C}{x_D-x_C}$.
$$m = \frac{-2-4}{2-(-2)} = \frac{-6}{4} = -1.5$$
3. Perpendicular gradient formula: $m_\perp = -\frac{1}{m}$.
$$m_\perp = -\frac{1}{-1.5} = \frac{2}{3} \approx 0.67$$
4. Point-slope form: $y - y_M = m_\perp(x - x_M)$.
$$y - 1 = \frac{2}{3}(x-0)$$
$$y = \frac{2}{3}x + 1$$

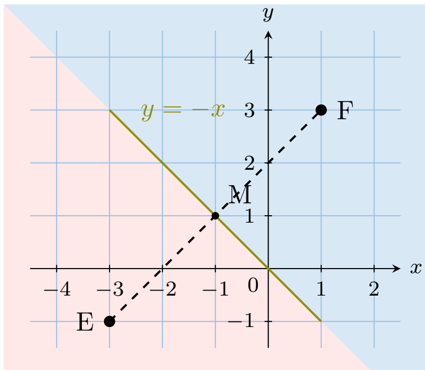


Ex 11: Consider two sites $E(-3,-1)$ and $F(1,3)$.

1. Calculate the coordinates of the midpoint M of the segment $[EF]$.
2. Calculate the gradient (slope) of the segment $[EF]$.
3. Determine the gradient of the perpendicular bisector.
4. Find the equation of the perpendicular bisector in the form $y = mx + c$.

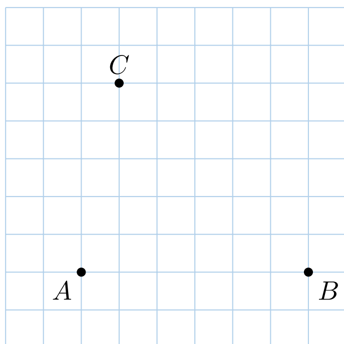
Answer:

1. Midpoint formula: $M\left(\frac{x_E+x_F}{2}, \frac{y_E+y_F}{2}\right)$.
$$M = \left(\frac{-3+1}{2}, \frac{-1+3}{2}\right) = (-1,1)$$
2. Gradient formula: $m = \frac{y_F-y_E}{x_F-x_E}$.
$$m = \frac{3-(-1)}{1-(-3)} = \frac{4}{4} = 1$$
3. Perpendicular gradient formula: $m_\perp = -\frac{1}{m}$.
$$m_\perp = -\frac{1}{1} = -1$$
4. Point-slope form: $y - y_M = m_\perp(x - x_M)$.
$$y - 1 = -1(x - (-1))$$
$$y - 1 = -x - 1$$
$$y = -x$$

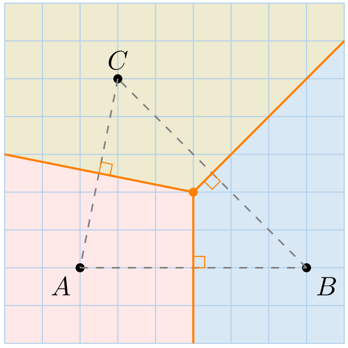


B.3 CONSTRUCTING VORONOI DIAGRAM FOR 3 SITES

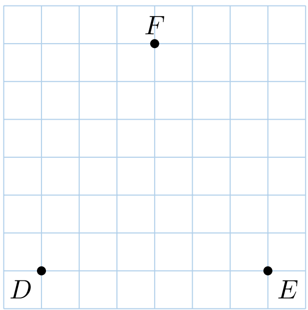
Ex 12: Using a ruler and a set square, draw the Voronoi diagram for three sites A, B, and C.



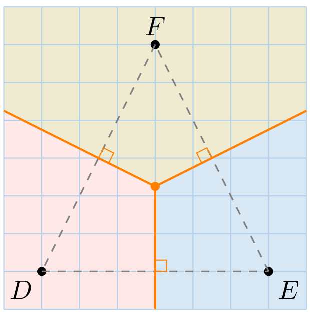
Answer: Draw the perpendicular bisectors of the segments $[AB]$, $[AC]$, and $[BC]$. The intersection point is the Voronoi vertex.



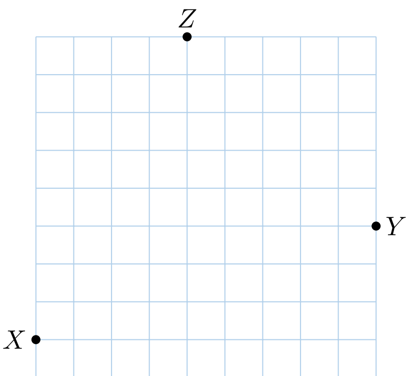
Ex 13: Using a ruler and a set square, draw the Voronoi diagram for three sites D, E, and F.



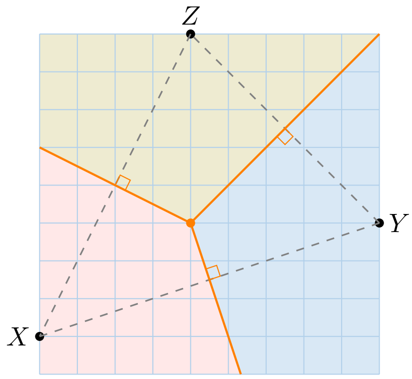
Answer: Draw the perpendicular bisectors of the segments $[DE]$, $[DF]$, and $[EF]$.



Ex 14: Using a ruler and a set square, draw the Voronoi diagram for three sites X, Y, and Z.



Answer: Draw the perpendicular bisectors of the segments $[XY]$, $[XZ]$, and $[YZ]$.



B.4 CONSTRUCTING VORONOI DIAGRAM WITH COORDINATES

Ex 15: Consider three sites $A(0,0)$, $B(6,0)$, and $C(2,4)$.
The perpendicular bisector of $[AB]$ has equation $x = 3$.

1. Find the equation of the perpendicular bisector of $[AC]$.
2. Find the coordinates of the Voronoi vertex V where the edges meet.

Answer:

1. **Bisector of $[AC]$:**
   - Midpoint of $[AC]$: $(\frac{0+2}{2}, \frac{0+4}{2}) = (1,2)$.
   - Gradient of $[AC]$: $m = \frac{4-0}{2-0} = 2$.
   - Gradient of bisector: $m_\perp = -\frac{1}{2} = -0.5$.
   - Equation:
   $$\begin{aligned} y - 2 &= -0.5(x-1) \\ y &= -0.5x + 0.5 + 2 \\ y &= -0.5x + 2.5 \end{aligned}$$
2. **Intersection (Vertex):**
   We have $x = 3$ and $y = -0.5x + 2.5$.
   Substitute $x = 3$:
   $$\begin{aligned} y &= -0.5(3) + 2.5 \\ &= -1.5 + 2.5 \\ &= 1 \end{aligned}$$
   Vertex $V(3,1)$.

Ex 16: Consider three sites $X(0,1)$, $Y(9,4)$, and $Z(4,9)$.

1. Find the equation of the perpendicular bisector of $[YZ]$.
2. Find the equation of the perpendicular bisector of $[XZ]$.
3. Find the coordinates of the Voronoi vertex V where the edges meet.

Answer:

1. **Bisector of $[YZ]$:**
   - Midpoint of $[YZ]$: $(\frac{9+4}{2}, \frac{4+9}{2}) = (6.5, 6.5)$.
   - Gradient of $[YZ]$: $m = \frac{9-4}{4-9} = \frac{5}{-5} = -1$.
   - Gradient of bisector: $m_\perp = -\frac{1}{-1} = 1$.
   - Equation:
   $$\begin{aligned} y - 6.5 &= 1(x - 6.5) \\ y &= x \end{aligned}$$

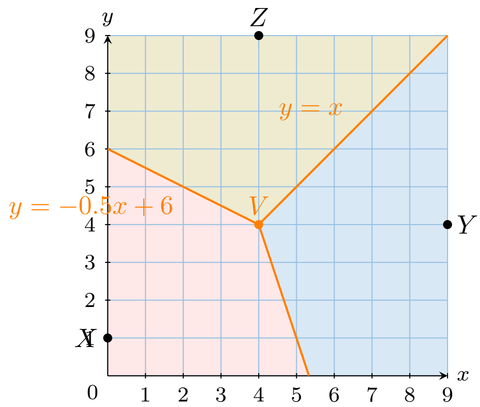
2. **Bisector of $[XZ]$:**
   - Midpoint of $[XZ]$: $(\frac{0+4}{2}, \frac{1+9}{2}) = (2, 5)$.
   - Gradient of $[XZ]$: $m = \frac{9-1}{4-0} = \frac{8}{4} = 2$.
   - Gradient of bisector: $m_\perp = -\frac{1}{2} = -0.5$.
   - Equation:

$$\begin{aligned} y - 5 &= -0.5(x - 2) \\ y &= -0.5x + 1 + 5 \\ y &= -0.5x + 6 \end{aligned}$$

3. **Intersection (Vertex):**
   We calculate the intersection of $y = x$ and $y = -0.5x + 6$. Substitute $y = x$:

$$\begin{aligned} x &= -0.5x + 6 \\ 1.5x &= 6 \\ x &= 4 \end{aligned}$$

   Since $y = x$, $y = 4$.
   Vertex $V(4, 4)$.



Ex 17: Consider three sites $A(3, 2)$, $B(8, 3)$, and $C(2, 7)$.

1. Find the equation of the perpendicular bisector of $[AB]$.
2. Find the equation of the perpendicular bisector of $[AC]$.
3. Find the coordinates of the Voronoi vertex V where the edges meet.

Answer:

1. **Bisector of $[AB]$:**
   - Midpoint of $[AB]$: $(\frac{3+8}{2}, \frac{2+3}{2}) = (5.5, 2.5)$.
   - Gradient of $[AB]$: $m = \frac{3-2}{8-3} = \frac{1}{5} = 0.2$.
   - Gradient of bisector: $m_\perp = -\frac{1}{0.2} = -5$.
   - Equation:

$$\begin{aligned} y - 2.5 &= -5(x - 5.5) \\ y &= -5x + 27.5 + 2.5 \\ y &= -5x + 30 \end{aligned}$$

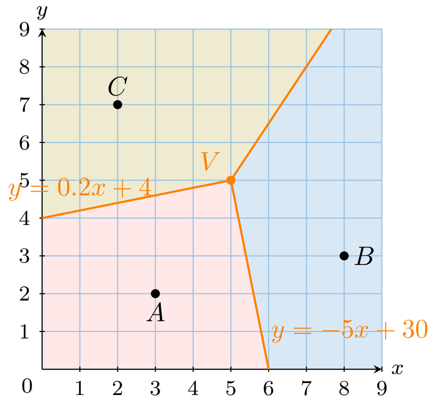
2. **Bisector of $[AC]$:**
   - Midpoint of $[AC]$: $(\frac{3+2}{2}, \frac{2+7}{2}) = (2.5, 4.5)$.
   - Gradient of $[AC]$: $m = \frac{7-2}{2-3} = \frac{5}{-1} = -5$.
   - Gradient of bisector: $m_\perp = -\frac{1}{-5} = 0.2$.
   - Equation:

$$\begin{aligned} y - 4.5 &= 0.2(x - 2.5) \\ y &= 0.2x - 0.5 + 4.5 \\ y &= 0.2x + 4 \end{aligned}$$

3. **Intersection (Vertex):**
   We find the intersection of $y = -5x + 30$ and $y = 0.2x + 4$.

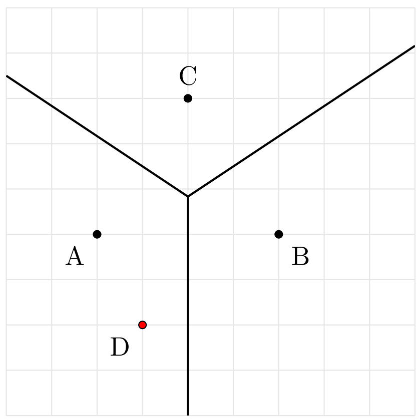
$$\begin{aligned} -5x + 30 &= 0.2x + 4 \\ 26 &= 5.2x \\ x &= 5 \end{aligned}$$

   Substitute $x = 5$: $y = 0.2(5) + 4 = 1 + 4 = 5$. Vertex $V(5, 5)$.



B.5 ADDING A NEW SITE

Ex 18: Site D is to be added to the Voronoi diagram shown below for sites A, B, and C.

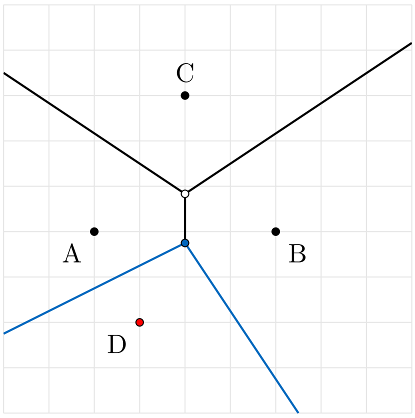


1. In which existing cell does site D lie?
2. Which of the existing cells will be affected by the introduction of site D?
3. Redraw the Voronoi diagram with site D added.

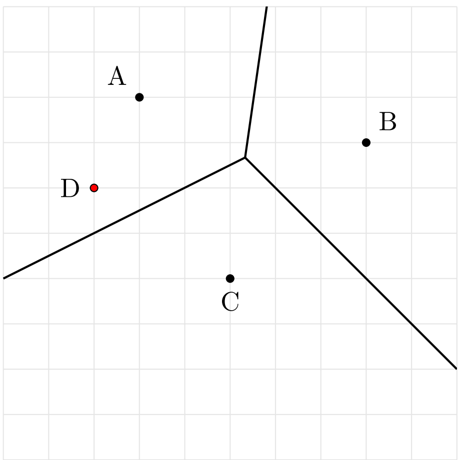
Answer:

1. **Locate Site D**: Visually, site D is closest to site A. It lies in the region of A.
2. **Identify Affected Cells**: Since D lands in cell A, cell A will definitely lose some area. D is also quite close to the boundary with B, so it is likely to steal some area from B as well. Cell C is far away and shielded by A, so it should remain unchanged.
3. **Construction**:

- **Step 1:** Draw the perpendicular bisector of AD.
- **Step 2:** Find where this new bisector intersects the existing edges of cell A. It intersects the edge between A and B. This creates a new vertex V.
- **Step 3:** From V, the boundary must now separate D and B. Draw the bisector of DB.
- **Step 4:** Erase the old edge segments that are now closer to D than to A or B.



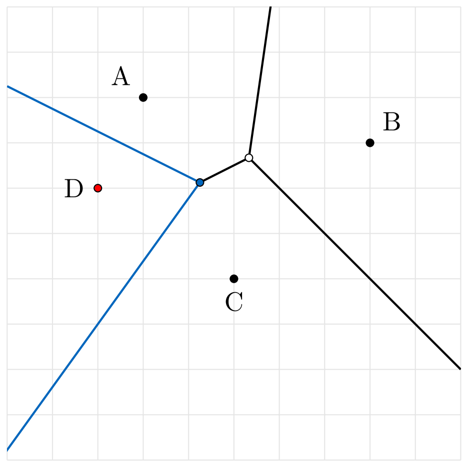
Ex 19: Site D is to be added to the Voronoi diagram shown below for sites A, B, and C. Note that triangle ABC is scalene (all sides are of different lengths).



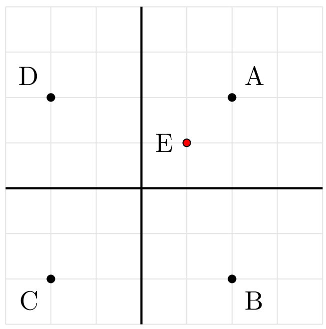
1. In which existing cell does site D lie?
2. Which of the existing cells will be affected by the introduction of site D?
3. Redraw the Voronoi diagram with site D added.

Answer:

1. **Locate Site D**: Visually, site $D(0,2)$ is much closer to site $A(1,4)$ than to C or B. It lies in the region of A.
2. **Identify Affected Cells**: Since D lands in cell A, cell A will definitely lose area. D is also positioned towards the border with C. It is likely that D will steal some area from C as well. Cell B is far to the right, shielded by A, and likely remains unaffected.
3. **Construction**:
   - **Step 1:** Draw the perpendicular bisector of AD. $A(1,4), D(0,2)$. Midpoint $(0.5,3)$. Slope $AD = 2$. Perp slope -0.5. Eq: $y = -0.5x + 3.25$.
   - **Step 2:** Find where this bisector intersects the existing edges of cell A. It meets the edge between A and C (line $y = 0.5x + 1$).
     - $-0.5x + 3.25 = 0.5x + 1 \Rightarrow x = 2.25$.
     - $y = 2.125$.
     - New Vertex $V_{new}(2.25, 2.125)$.
   - **Step 3:** From V_{new}, the boundary must separate D and C. Draw the bisector of DC.
   - **Step 4:** Remove the old edge segments that are now closer to D.



Ex 20: Site E is to be added to the Voronoi diagram shown below for sites A, B, C, and D.



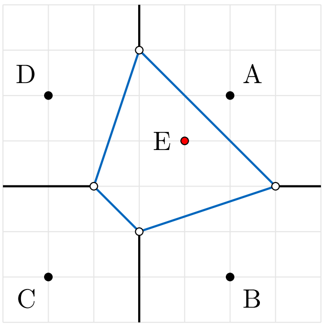
1. In which existing cell does site E lie?
2. Explain why site E will share a boundary with all other sites (A, B, C, D).
3. Redraw the Voronoi diagram with site E added.

Answer:

1. **Locate Site E**:
   Visually, site $E(3,3)$ is closest to site A. It lies in the region of A.
2. **Impact on other cells**:
   Since E is closer to the vertex V than the original sites, the new region for E will cover V. Since V was the meeting point of regions A, B, C, D, the new region E will now separate them and share a boundary with each.
3. **Construction**:
   We calculate the new vertices by intersecting the bisectors of E and its neighbors with the existing boundaries.
   - **Vertex between A, D, E**: Bisector AE intersects AD boundary at $(2,5)$.

- **Vertex between A, B, E**: Bisector AE intersects AB boundary at $(5, 2)$.
- **Vertex between B, C, E**: Bisector BE intersects BC boundary at $(2, 1)$.
- **Vertex between C, D, E**: Bisector DE intersects CD boundary at $(1, 2)$.

Connect these points to form the cell for E.



C NEAREST NEIGHBOR INTERPOLATION

C.1 INTERPOLATING USING NEAREST NEIGHBOR ALGORITHM

Ex 21: A city has weather stations at $A(1, 1)$, $B(5, 1)$, and $C(3, 5)$.
The temperatures recorded are: $A : 20°\text{C}$, $B : 22°\text{C}$, $C : 18°\text{C}$.
Estimate the temperature at point $P(2, 2)$ using nearest neighbor interpolation.

1. Calculate PA^2, PB^2, and PC^2.
2. Which region contains P?
3. What is the estimated temperature?

Answer: The squared distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

1. - $PA^2 = (2-1)^2 + (2-1)^2 = 1^2 + 1^2 = 2$.
   - $PB^2 = (2-5)^2 + (2-1)^2 = (-3)^2 + 1^2 = 9 + 1 = 10$.
   - $PC^2 = (2-3)^2 + (2-5)^2 = (-1)^2 + (-3)^2 = 1 + 9 = 10$.
2. Since PA^2 is the smallest value $(2 < 10)$, point P is closest to site A. Therefore, P is in region A.
3. Estimated temperature = Temperature at $A = 20°\text{C}$.

Ex 22: Agricultural scientists measure the weekly rainfall (in mm) at three stations $R(1, 5)$, $S(5, 5)$, and $T(3, 1)$.
The recorded values are: $R : 12$ mm, $S : 8$ mm, $T : 15$ mm.
Estimate the rainfall at a farm located at $P(4, 4)$ using nearest neighbor interpolation.

1. Calculate PR^2, PS^2, and PT^2.
2. Which region contains P?
3. What is the estimated rainfall?

Answer: The squared distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

1. - $PR^2 = (4-1)^2 + (4-5)^2 = 3^2 + (-1)^2 = 9 + 1 = 10$.
   - $PS^2 = (4-5)^2 + (4-5)^2 = (-1)^2 + (-1)^2 = 1 + 1 = 2$.
   - $PT^2 = (4-3)^2 + (4-1)^2 = 1^2 + 3^2 = 1 + 9 = 10$.
2. Since PS^2 is the smallest value $(2 < 10)$, point P is closest to site S. Therefore, P is in region S.
3. Estimated rainfall = Rainfall at $S = 8$ mm.

Ex 23: A biologist measures the soil pH at three locations $D(-2, 3)$, $E(2, 4)$, and $F(0, -1)$.
The pH levels are: $D : 6.5$, $E : 7.2$, $F : 5.8$.
Estimate the pH level at location $P(1, 2)$ using nearest neighbor interpolation.

1. Calculate PD^2, PE^2, and PF^2.
2. Which region contains P?
3. What is the estimated pH level?

Answer: The squared distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

1. - $PD^2 = (1-(-2))^2 + (2-3)^2 = 3^2 + (-1)^2 = 9 + 1 = 10$.
   - $PE^2 = (1-2)^2 + (2-4)^2 = (-1)^2 + (-2)^2 = 1 + 4 = 5$.
   - $PF^2 = (1-0)^2 + (2-(-1))^2 = 1^2 + 3^2 = 1 + 9 = 10$.
2. Since PE^2 is the smallest value $(5 < 10)$, point P is closest to site E. Therefore, P is in region E.
3. Estimated pH level = pH at $E = 7.2$.

D THE TOXIC DUMP PROBLEM

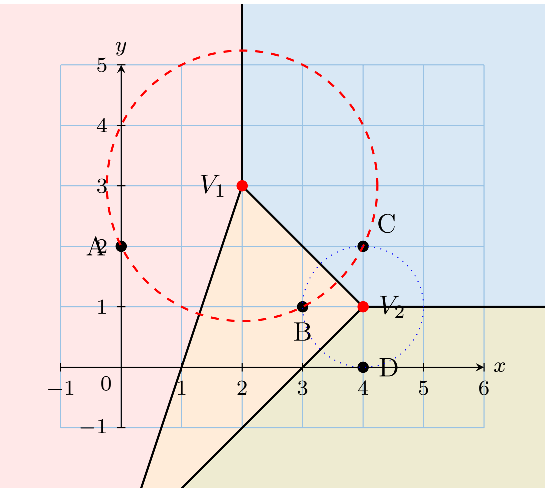
D.1 OPTIMIZING LOCATIONS

Ex 24: The Voronoi diagram for four towns A, B, C, and D has two internal vertices located at $V_1(2, 3)$ and $V_2(4, 1)$. The coordinates of the towns are $A(0, 2)$, $B(3, 1)$, $C(4, 2)$, and $D(4, 0)$.
We want to locate a toxic dump as far as possible from any town.

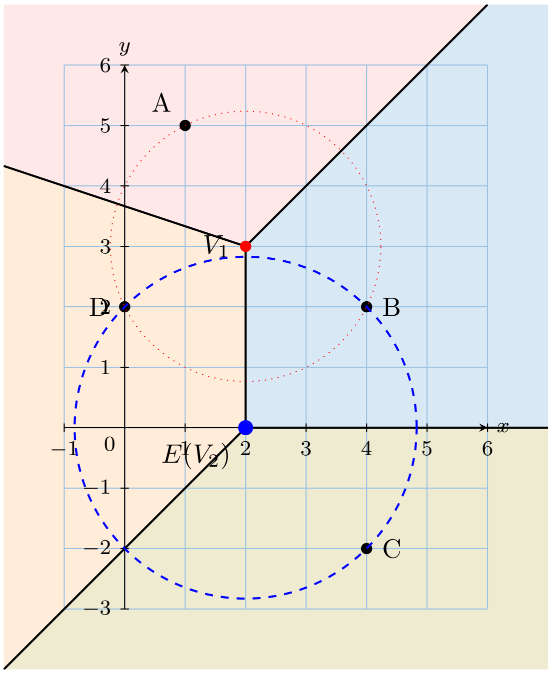
1. Calculate the distance from vertex V_1 to its nearest site $A(0, 2)$.
2. Calculate the distance from vertex V_2 to its nearest site $D(4, 0)$.
3. Which of these two vertices is the best location for the toxic dump? What is the radius of the empty circle there?

Answer:

1. Distance from $V_1(2, 3)$ to $A(0, 2)$:
$$d(V_1, A) = \sqrt{(2-0)^2 + (3-2)^2} \approx 2.24$$
(Note: V_1 is equidistant from A, B, C).
2. Distance from $V_2(4, 1)$ to $D(4, 0)$:
$$d(V_2, D) = \sqrt{(4-4)^2 + (1-0)^2} = 1$$
(Note: V_2 is equidistant from B, C, D).
3. Comparing the distances, $2.24 > 1$. The vertex V_1 is further from the towns than V_2.
The best location is V_1, and the radius of the largest empty circle is $\sqrt{5} \approx 2.24$.



Ex 25: A city currently has four fire stations located at $A(1,5)$, $B(4,2)$, $C(4,-2)$, and $D(0,2)$.
The city council wants to build a **fifth fire station**. To maximize efficiency and coverage, the new station should be located at a point that is as far as possible from the existing stations (filling the largest gap in coverage).
The Voronoi diagram for the current stations has two internal vertices located at $V_1(2,3)$ and $V_2(2,0)$.



1. Calculate the distance from vertex V_1 to its nearest station A.

2. Calculate the distance from vertex V_2 to its nearest station D.

3. Determine the coordinates of the optimal location for the new fire station E. Justify your answer.

Answer: The optimal location corresponds to the center of the Largest Empty Circle, which is the Voronoi vertex furthest from the sites.

1. Distance from $V_1(2,3)$ to $A(1,5)$:

$$d(V_1, A) = \sqrt{(2-1)^2 + (3-5)^2} = \sqrt{5} \approx 2.24$$

(Note: V_1 is equidistant from A, B, D).

2. Distance from $V_2(2,0)$ to $D(0,2)$:

$$d(V_2, D) = \sqrt{(2-0)^2 + (0-2)^2} = \sqrt{8} \approx 2.83$$

(Note: V_2 is equidistant from B, C, D).

3. **Comparison:**
Since $\sqrt{8} > \sqrt{5}$ $(2.83 > 2.24)$, the gap around V_2 is larger than the gap around V_1.
To maximize coverage, the new station E should be built at **$V_2(2,0)$**.